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Information in, on, and from the Spacetime

by

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Abstract

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Professor Raphael Bousso, Chair

Information theory has become increasingly more prominent in physics research, and it is no different in gravity. In this thesis we look at several connections between information and the spacetime.

Motivated in part by the black-hole information paradox, we complete a proof of an asymptotic entropy bound and consider the equivalence principle, both in attempt to answer where information may hide IN asymptotically flat space-times. We consider information bounds IN Riemann flat empy regions and ON the null conformal boundary. Our results led us to investigate the physical principles that render potential information at infinity unobservable and consider a communication protocol illustrating a connection between information and energy. We were able to show that quantum effects can account for unobservability of asymptotic charges.

Holography is the idea that all the information about a spacetime is contained ON a lower dimensional surface (usually the boundary). Its most concrete realization, AdS/CFT, has already revolutionized physics. Still, that has not stopped physicists from seeking suitable generalizations, and it is in this context that holographic screens were introduced. The causal future of a spacetime region, is the subset of the spacetime where information from that region can reach. A certain characterization of the boundary of the future was needed for proving a new area theorem for holographic screens. We provide a proof of this characterization.

Entanglement entropy provides a way of quantifying how information is spread in a quantum system. However, entanglement entropy alone does not provide an exhaustive picture of the distribution of information in a quantum system, and so several other entanglement measures have been studied, including, for instance, entenglement of distillation, entanglement of formation, and entenglament of purification. In many cases, holography provides an efficient way of studying entanglement measures. This allows us to learn about information properties of quantum systems FROM the spacetime. We study potential holographic duals for entanglement of purification and its multipartite generalization, and prove several of their properties.

To my grandmother Leja Sapirstein Feiman (03/15/1919 – 05/17/2019)

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Chapter 1

Introduction

Known entropy bounds, and the Generalized Second Law, were recently shown to imply bounds on the information arriving at future null infinity. We complete this derivation by including the contribution from gravitons. We test the bounds in classical settings with gravity and no matter. In Minkowski space, the bounds vanish on any subregion of the future boundary, independently of coordinate choices. More generally, the bounds vanish in regions where no gravitational radiation arrives. In regions that do contain Bondi news, the bounds are compatible with the presence of information, including the information stored in gravitational memory. All of our results are consistent with the equivalence principle, which states that empty Riemann-flat spacetime regions contain no classical information. We also discuss the possibility that Minkowski space has an infinite vacuum degeneracy labeled by a choice of Bondi coordinates (a classical parameter, if physical). We argue that this degeneracy cannot have any observational consequences if the equivalence principle holds. Our bounds are consistent with this conclusion. This all appears in Chapter 2, which is based on [32]

To study quantum gravity in asymptotically flat spacetimes, one would like to understand the algebra of observables at null infinity. Here we show that the Bondi mass cannot be observed in finite retarded time, and so is not contained in the algebra on any finite portion of $\mathscr{I}^+$. This follows immediately from recently discovered asymptotic entropy bounds. We verify this explicitly, and we find that attempts to measure a conserved charge at arbitrarily large radius in fixed retarded time are thwarted by quantum fluctuations. We comment on the implications of our results to flat space holography and the BMS charges at $\mathscr{I}^+$. This can be seen in Chapter 3, which is based on [33].

In Chapter 4, based on [4], we prove that the boundary of the future of a surface K consists precisely of the points p that lie on a null geodesic orthogonal to K such that between K and p there are no points conjugate to K nor intersections with another such geodesic. Our theorem has applications to holographic screens and their associated light sheets and in particular enters the proof that holographic screens satisfy an area law.

We study the conjectured holographic duality between entanglement of purification and the entanglement wedge cross-section in Chapter 5, which is based on [12]. There, we generalize both quantities and prove several information theoretic inequalities involving them. These include upper bounds on conditional mutual information and tripartite information, as well as a lower bound for tripartite information. These inequalities are proven both holographically and for general quantum states. In addition, we use the cyclic entropy inequalities to derive a new holographic inequality for the entanglement wedge cross-section, and provide numerical evidence that the corresponding inequality for the entanglement of purification may be true in general. Finally, we use intuition from bit threads to extend the conjecture to holographic duals of suboptimal purifications.

In Chapter 6, based on [11], we generalize the entanglement of purification and its conjectured holographic dual to conditional and multipartite versions of the same, where the optimization defining the entanglement of purification is now optimized in either a constrained way or over multiple parties. We separately derive new constraints on both the conditional entanglement of purification and its conjectured holographic dual object that match, further reinforcing the likelihood of this conjecture. We also show that the multipartite objects we define, despite obeying several of the same inequalities, are not holographic duals of each other. Further, we find inequalities that are true only for the bulk objects, and thus could provide additional consistency checks for states dual to (semi)-classical bulk geometries.

Appendices provide supplemental examples and calculations.

Chapter 2

Information Content of Gravitational Radiation and the Vacuum

2.1 Introduction

Entropy bounds control the information flow through any light-sheet [24], in terms of the area difference between two cuts σ_1, σ_2 of the light-sheet:

$$S \leq \frac{A[\sigma_1] - A[\sigma_2]}{4G\hbar} \ . \tag{2.1}$$

A light-sheet is a null hypersurface consisting of null geodesics orthogonal to σ_1 that are nowhere expanding. A cut is a spatial cross-section of the light-sheet.

In simple settings, one can take S to be the thermodynamic entropy of isolated systems crossing the light-sheet. More generally, the definition of S is subtle, because in field theory there are divergent contributions from vacuum entanglement across σ_1 and σ_2. Precise definitions of S were found only recently, leading to rigorous proofs of two different field theory limits ($G \to 0$) of Eq. (2.1). The proofs apply to free [35, 36] and interacting [34] scalar fields. Entropy bounds have also been verified [34] or proven [70] holographically for interacting gauge fields with a gravity dual.

Gravitational waves heat water, so they can be used to send information. In general, it is challenging to distinguish between gravitational waves and a curved spacetime. This can be done approximately in a setting where the wavelength of the radiation is small compared to other curvature radii in the geometry. A more rigorous notion of gravitational radiation is the "Bondi news," which is defined in terms of an asymptotic expansion of the metric of asymptotically flat spacetimes [22, 90].

The Bondi news corresponds to gravitational radiation that reaches distant regions (see Fig. 2.1). It has been observed by monitoring test masses far from the source [3, 2]. Its definition contains a rescaling by a factor of the radius, so that it remains finite as the radiation is diluted and weakened. Ultimately, it can be thought of as a spin-2 degree of freedom on future null infinity, $\mathcal{I}^+$.

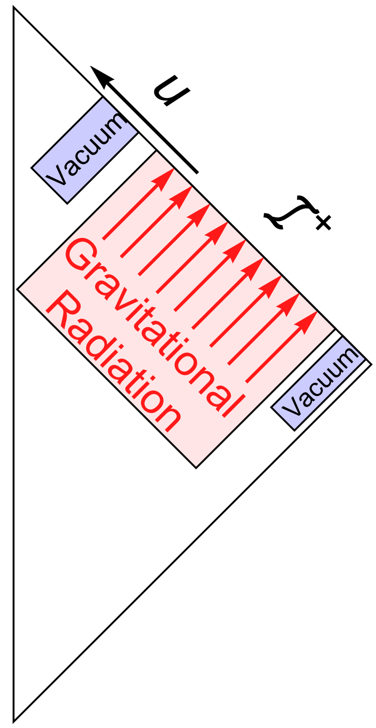


Figure 2.1: Penrose diagram of an asymptotic flat spacetime. Gravitational radiation (i.e., Bondi news) arrives on a bounded portion of $\mathcal{I}^+$ (red). The asymptotic regions before and after this burst (blue) are Riemann flat. The equivalence principle requires that observers with access only to the flat regions cannot extract classical information; however, an observer with access to the Bondi news can receive information (see Sec. 2.3). We find in Sec. 2.4 that the asymptotic entropy bounds of Sec. 2.2 are consistent with these conclusions.

Recently, bounds on the entropy of arbitrary subregions of $\mathcal{I}^+$ were obtained as the limit of known bulk entropy bounds [25]. These bounds constrain both the vacuum-subtracted entropy of states reduced to the subregion, and its derivatives as the subregion is varied. However, only nongravitational fields were treated rigorously. In this paper, we show how to incorporate gravitational radiation into the asymptotic entropy bounds.

The bulk entropy bounds that formed the starting point of Ref. [25] have been proven for certain fields [101, 35, 34, 36, 70]. Unless there is a discontinuity in the asymptotic limit, we expect these proofs to apply to the asymptotic bounds as well. Explicit proofs have not yet been given for a spin-2 field, however. To be conservative, the asymptotic bounds on gravitational radiation should be regarded as a conjecture.

Therefore, we will perform a simple consistency check: we ask whether the bounds are compatible with the equivalence principle. We take this principle to be the statement that an empty, Riemann-flat spacetime region contains no classical information. (By this we mean a subset of Minkowski space, not of a Riemann-flat spacetime with nontrivial topology. In this paper, "flat" will always mean Riemann-flat and devoid of matter.) In particular, the classical information of the spacetime geometry is contained only in its Riemann curvature, and not, for example, in the choice of coordinates.

The simplest setting is empty Minkowski space. In any subregion of $\mathcal{I}^+$, our upper bounds vanish, implying that the vacuum-subtracted entropy is nonpositive and independent of the subregion. (In particular, the upper bounds do not depend on a "choice of vacuum" of Minkowski space.) This is consistent with the equivalence principle, which tells us that no classical information is present.

The asymptotic metric of Minkowski space, written in Bondi coordinates, is not uniquely fixed by fall-off conditions. One can freely choose the $1/r$ correction to the shape of spheres specified by setting the coordinates u and r to constants. (Note that this correction describes the shape of an embedded surface, whose location is determined by an arbitrary coordinate choice. Its shape is not indicative of any actual curvature of Minkowski space, which is manifestly Riemann-flat.) The freedom corresponds to a choice of a single real function $c(\Omega)$ of the coordinates on the sphere. For a brief review on Bondi frames, see Appendix A.

Recently, this degeneracy in the choice of Bondi coordinates has been interpreted as a degeneracy of the actual vacuum state of Minkowski space [92, 58]. We take no position on the formal convenience of elevating a classical coordinate choice to a degeneracy of the vacuum.

However, the equivalence principle rules out the possibility that a coordinate choice in Minkowski space has any measurable consequences. Therefore, $c(\Omega)$ must be unobservable. This is consistent with the fact that our bounds are insensitive to $c(\Omega)$ and vanish identically in Minkowski space.

We also consider a classical gravitational wavepacket with finite support, which arrives at $\mathcal{I}^+$ as Bondi news. In portions of $\mathcal{I}^+$ where the news has no support, our upper bounds vanish. This is consistent with the absence of classical information according to the equivalence principle: distant regions without gravitational radiation are Riemann-flat, so their geometry cannot be distinguished from Minkowski space.

In Bondi coordinates, the Bondi news does change the function $c(\Omega)$, by an integral of the news [92, 93, 58]. Since the news can be measured, this integral can be measured; for example, it results in a permanent displacement of physical detectors. Thus, in a nonvacuum spacetime, $c(\Omega)$ is a coordinate choice only in that it can be picked freely *either* before *or* after the burst. The difference—the gravitational memory—is invariant and physical. The equivalence principle, and our bounds, constrain *how* the memory can be observed: namely, only by recording the news with physical detectors (which must be present during the burst). The memory cannot be measured by merely probing the asymptotic vacuum regions before and after the burst.

Outline In Sec. 2.2, we review the derivation of asymptotic entropy bounds of Ref. [25] (Sec. 2.2), and we show that they respond to gravitational radiation through the square of the Bondi news (Sec. 2.2).

In Sec. 2.3, we discuss implications of the equivalence principle. In Sec. 2.3, we consider the term $C_{AB}(\infty)$ that appears in the asymptotic (Bondi) metric of asymptotically flat spacetimes. This term can be nonvanishing even in Minkowski space and has been interpreted as labelling degenerate vacua [92, 7, 58]. Since it corresponds to a coordinate choice in Minkowski space, the equivalence principle demands that C_{AB} be unobservable in any experiment. In Sec. 2.3, we consider gravitational memory (the integral of Bondi news). The equivalence principle implies that the memory can only be measured by an observer or apparatus that has access to all the Bondi news that produces the memory. In Sec. 2.3, we discuss "soft" gravitons and gravitational waves, by which we mean waves with long wavelength compared to some other time scale in the process that produces them. Given enough time, such excitations can be distinguished from the vacuum and so their information content is unconstrained by the equivalence principle.

In Sec. 2.4, we discuss implications of the entropy bounds of Sec. 2.2, in the same settings considered in Sec. 2.3. In Minkowski space, there is no news, and all our upper bounds vanish. We also consider a classical probabilistic ensemble (i.e., a mixed state) of classical gravitational wave bursts. We find that our bounds permit an observer to distinguish between different classical messages if and only if the observer has access to the news. Thus the implications of our entropy bounds are consistent with the conclusions we draw from the equivalence principle in Sec. 2.3.

In Appendix B, we discuss an asymptotic entropy bound proposed by Kapec *et al.* [67]. We focus on the case of empty Minkowski space. Whether this bound differs from (a special case of) ours depends on the definition of the entropy, which was not fully specified in Ref. [67]. We argue that consistency with the equivalence principle requires a choice under which the bounds agree. We clarify that the extra term in the upper bound of Ref. [67] originates from a difference in how the relevant null surfaces are constructed before the asymptotic limit is taken.

In Appendix C, we apply the bounds of Sec. 2.2 to a single graviton wavepacket. This case is not obviously constrained by the equivalence principle and so lies outside the main line of argument pursued here. We find that our bounds have implications similar to those derived for the classical Bondi news in Sec. 2.4.

2.2 Asymptotic Entropy Bounds and Bondi News

In Ref. [25], entropy bounds were applied to a distant planar light-sheet. The bounds can be expressed in terms of the stress tensor of matter crossing the light-sheet, and the square of the shear of the light-sheet. It was shown that the matter contribution is independent of the orientation of the light-sheet in the asymptotic limit. However this was not proven for the contribution from the shear. Here we fill this gap by demonstrating that the shear term contributes to the upper bounds as the square of the Bondi news. Thus it is associated with gravitational radiation reaching the boundary. In particular, this implies that the asymptotic bounds of Ref. [25] are fully independent of the orientation of the light-sheets used to derive them.

Asymptotic Entropy Bounds

In this subsection we briefly review the derivation and formulation of the asymptotic entropy bounds of Ref. [25]. Expectation value brackets are left implicit throughout.

We consider entropy bounds [24, 47, 37] in the general form of Eq. (2.1). In the weak-gravity limit, Newton's constant G is taken to become small, and a light-sheet is chosen that consists of initially parallel light-rays ($\theta_0 = 0$). An example of this is a null plane $t - z = \text{const}$ in Minkowski space. The effects of matter on the light-sheet are computed to leading nontrivial order in G, from the focussing equation [100]

$$-\frac{d\theta}{dw} = 8\pi G\, T_{ab} k^a k^b + \varsigma_{ab}\varsigma^{ab} \ . \tag{2.2}$$

Here T_{ab} is the matter stress tensor, k^a is the tangent vector to the light-rays that comprise the light-sheet, w is an affine parameter and ς is the shear (defined by Eq. 2.20). The expansion θ is the logarithmic derivative of the area of a cross-section spanned by infinitesimally nearby light-rays.

By Eq. (2.1), the upper bound is given by the total area loss between two cross-sections of the light-sheet. It can be computed by integrating Eq. (2.2) twice along the light-rays, and then across the transverse directions. If the shear scales as $G^{1/2}$, the area loss will scale as G, so Newton's constant drops out in Eq. (2.1). The resulting bound involves only Planck's constant $\hbar$, so it can be viewed as a pure field theory statement.

Near the boundary of an asymptotically flat spacetime, the matter stress tensor falls off as r^{-2} and the shear associated with gravitational radiation falls of as r^{-1}, so the above argument can be carried out at finite G, as an expansion in G/r^2. In particular, one can work on a Minkowski background,

$$ds^2 = -du^2 - 2du\, dr + r^2 d\Omega^2 \ , \tag{2.3}$$

and compute area differences at order G/r^2, by integrating the focussing equation (2.2).

Keeping the radiation under consideration fixed, the area of the radiation front increases in the asymptotic limit as the local stress tensor decreases. It is convenient to rescale both [25], and formulate asymptotic entropy bounds directly in terms of finite quantities on $\mathcal{I}^+$. The asymptotic energy flux is the energy arriving on $\mathcal{I}^+$ per unit advanced time and unit solid angle:[1]

$$\hat{\mathcal{T}} = \hat{T}_{uu} + \hat{\varsigma}_{ab}\hat{\varsigma}^{ab} \ , \tag{2.4}$$

The first term is the energy flux of nongravitational radiation,

$$\hat{T}_{uu} = \lim_{r\to\infty} r^2 T_{uu} \ , \tag{2.5}$$

[1] We will generally refer to boundary versions of bulk quantities by adding a hat.

The second term is set by the shear of the light-sheet and will be defined in Sec. 2.2. It will be shown to correspond to the energy delivered by gravitational waves.

In Ref. [25], the basic tool for deriving the asymptotic entropy bounds is the notion of a distant planar light-sheet. Let $p \in \mathcal{I}^+$ be a point at affine time u_p and angle $\vartheta_p = \pi$. Let $H(u_p)$ be the boundary of the past of p:

$$H(u_p) \equiv \dot{I}^-(p) \ , \quad p \in \mathcal{I}^+ \ . \tag{2.6}$$

As discussed in Ref. [25], $H(u_p)$ is a null hypersurface. At $O(G/u_p^2)^0$, it is the null plane $t + z = u_p$ in Minkowski space, with affine parameter $w \equiv t - z$ and tangent vector $k^\mu = dx^\mu/dw$. In the $\{u, r, \vartheta, \phi\}$ coordinates, k^μ has components

$$k^u = \cos^2(\vartheta/2) \tag{2.7}$$
$$k^r = -(\cos\vartheta)/2 \tag{2.8}$$
$$k^\vartheta = \sin\vartheta\cos^2(\vartheta/2)/(u_p - u) \tag{2.9}$$
$$k^\phi = 0 \ . \tag{2.10}$$

In Ref. [25] it was shown that a number of known weak-gravity entropy bounds apply on $H(u_p)$. Cuts on different $H(u_p)$ were identified for different u_p by using the same function $u(\Omega)$ to define each cut; this function also defines a cut on $\mathcal{I}^+$. Bulk entropy bounds were applied to subregions defined by the cuts. The limit as $u_p \to \infty$ was taken and the bulk entropy bounds were re-expressed in terms of the asymptotic energy flux $\hat{\mathcal{T}}$. We will now list these results; see Ref. [25] for details.

From the Quantum Null Energy Condition [37, 36] (QNEC) on $H(u_p)$, one obtains the Boundary QNEC,

$$\frac{1}{\delta\Omega}\frac{d^2}{du^2}\hat{S}_{\rm out}[\hat{\sigma}, \Omega] \le \frac{2\pi}{\hbar}\,\hat{\mathcal{T}} \ . \tag{2.11}$$

Here $\delta\Omega$ is a small solid angle element near a null geodesic at angle Ω on $\mathcal{I}^+$. The second derivative is computed as this element is pushed to larger u, starting a given cut $\hat{\sigma}$ of $\mathcal{I}^+$. The limit as $\delta\Omega \to 0$ is implicit. The entropy $\hat{S}_{\rm out}$ is the von Neumann entropy of the state of the subregion of $\mathcal{I}^+$ above the cut. That is,

$$\hat{S}_{out} \equiv -\mathrm{tr}_{>\hat{\sigma}}\,\rho\log\rho \ . \tag{2.12}$$

The reduced state in the region above the cut $\hat{\sigma}$ is defined by

$$\rho = \mathrm{tr}_{<\hat{\sigma}}\,\rho_g \ , \tag{2.13}$$

where ρ_g is the global state on $\mathcal{I}^+$. We need not include future timelike infinite since we assume that all matter decays to radiation at sufficiently late times. Note that all cuts of $\mathcal{I}^+$ have the same intrinsic and extrinsic geometry. Therefore, divergent terms in $S_{\rm out}$ drop out when differences are computed, or when derivatives are taken (also below). With the above definition, the QNEC has been proven for free scalar fields, and also for interacting gauge fields with a gravity dual [70].

From the differential, weak gravity Generalized Second Law (GSL) on $H(u_p)$ [20, 101], or by integrating Eq. (2.11), one obtains the Boundary GSL in differential form

$$-\frac{1}{\delta\Omega}\frac{d}{du}\hat{S}_{\rm out}[\hat{\sigma};\Omega] \leq \frac{2\pi}{\hbar}\int_{\hat{\sigma}}^{\infty} du\, \hat{\mathcal{T}}\ . \tag{2.14}$$

From the integrated weak-gravity GSL on $H(u_p)$, or by integrating Eq. (2.14), one obtains the Boundary GSL in integral form,

$$\hat{S}_{\rm out}[\hat{\sigma}_2] - \hat{S}_{\rm out}[\infty] \leq \frac{2\pi}{\hbar}\int_{\hat{\sigma}_2}^{\infty} d^2\Omega\, du\, [u - u_2(\Omega)]\, \hat{\mathcal{T}}\ , \tag{2.15}$$

where $\hat{S}_{\rm out}[\infty]$ is to be understood in a limiting sense.

Finally, from the Quantum Bousso Bound (QBB) [35, 34] on finite "slabs" of $H(u_p)$ one obtains a Boundary QBB,

$$\hat{S}_C[\hat{\sigma}_1, \hat{\sigma}_2] \leq \frac{2\pi}{\hbar}\int_{\hat{\sigma}_2}^{\hat{\sigma}_1} d^2\Omega\, du\, \hat{g}(u)\, \hat{\mathcal{T}}(u,\Omega)\ . \tag{2.16}$$

The weighting function $\hat{g}$ is different for free and interacting bulk fields [35, 34]. Since fields become free asymptotically, we expect that it is given by the free field expression $\hat{g}(u) = (u_1 - u)(u - u_2)/(u_1 - u_2)$.

In Eq. (2.16), $\hat{S}_C$ is the vacuum-subtracted entropy [74, 38] of a finite affine interval on $\mathcal{I}^+$. It is defined directly on the finite portion of the light-sheet between $\hat{\sigma}_1$ and $\hat{\sigma}_2$, as the difference of two von Neumann entropies

$$\hat{S}_C[\hat{\sigma}_1, \hat{\sigma}_2] = -\mathrm{tr}\,\rho \log\rho + \mathrm{tr}\,\chi\log\chi\ . \tag{2.17}$$

Here the density operator ρ is obtained from the global quantum state ρ_g by tracing out the exterior of the region between $\hat{\sigma}_1$ and $\hat{\sigma}_2$; and χ is similarly obtained from the global vacuum state.[2] The ultraviolet contributions from vacuum entanglement are the same in both reduced states, so they cancel out [74, 38, 35]. With this definition, the QBB has been proven both for free and interacting scalar fields. It has also been verified for gauge fields with gravity duals [34].

For free theories, the algebra of operators factorizes over the null geodesics that generate the light-sheet [101]. We expect that this case applies to $\mathcal{I}^+$. Then the von Neumann entropy of the vacuum state restricted to the semi-infinite region above a cut $\hat{\sigma}$ is independent of the cut. Therefore, we have

$$\hat{S}_{\rm out}[\hat{\sigma}_2] - \hat{S}_{\rm out}[\hat{\sigma}_1] = \hat{S}_C[\hat{\sigma}_2] - \hat{S}_C[\hat{\sigma}_1]\ , \tag{2.18}$$

[2] As discussed in the introduction, the equivalence principle requires that the reduced vacuum state is unique, so it implies that the definition of the vacuum-subtracted entropy is unambiguous. We return to this point in App. B.

where $\hat{S}_C[\hat{\sigma}]$ is now computed on the semi-infinite regions above $\hat{\sigma}_1$ and $\hat{\sigma}_2$. Thus we can also express other bounds, Eqs. (2.11) and (2.14), in terms of derivatives and differences of the manifestly finite quantity $\hat{S}_C$, instead of $\hat{S}_{\rm out}$.

In particular, we can write the integrated Boundary GSL, Eq. (2.15), as

$$\hat{S}_C[\hat{\sigma}_2] \leq \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega\, du\, [u - u_2(\Omega)]\, \hat{\mathcal{T}} \ . \tag{2.19}$$

We have used the fact that the reduced density matrix of any physical state above a cut at sufficiently large u is that of the vacuum restricted to the same region, and thus $\hat{S}_C[\infty] = 0$. We will use this form of the integrated Boundary GSL in Sec. 2.4.

Bondi News as Shear on Distant Light-Sheets

Let ς_{ab} be the shear tensor on $H(u_p)$, defined as the tracefree part of the extrinsic curvature:

$$\varsigma_{ab} = B_{ab} - \frac{1}{2}\theta q_{ab} \ , \tag{2.20}$$

where $B_{ab} = q_a{}^c q_b{}^d \nabla_c k_d$, and q_{ab} is the metric on the cuts $w =$ const. One could choose different cuts, but some foliation of $H(u_p)$ into cuts has to be chosen in order to discuss the evolution of the shear. The shear tensor has only transverse components, so its information is fully captured by the lower-dimensional tensor

$$\varsigma_{\bar{A}\bar{B}} \equiv \varsigma_{ab} e^a{}_{\bar{A}} e^b{}_{\bar{B}} \ . \tag{2.21}$$

The $D-2$ orthonormal vectors $e^a{}_{\bar{A}}$ are tangent to the cut. Below we will denote any projection with the $e^a{}_{\bar{A}}$ by capital indices placed on higher-dimensional tensors.

The evolution equation for the shear is [100, 85]

$$\frac{d}{dw}\varsigma_{\bar{A}\bar{B}} = W_{\bar{A}\bar{B}} - \theta\, \varsigma_{\bar{A}\bar{B}} \ , \tag{2.22}$$

where

$$W_{\bar{A}\bar{B}} = -\, C_{abcd} e^a{}_{\bar{A}} k^b e^c{}_{\bar{B}} k^d \equiv -\, C_{\bar{A}b\bar{B}d} k^b k^d \tag{2.23}$$

and C_{abcd} is the Weyl tensor. We now recall that at fixed (u, Ω) there is no difference between expansions in inverse powers of u_p and r, since [25]

$$r = \frac{u_p - u}{2\cos^2(\vartheta/2)} \ . \tag{2.24}$$

The asymptotic behavior of the Weyl tensor is [49]

$$\begin{aligned}
C_{\bar{A}u\bar{B}\vartheta} &\sim O(r^{-1}) \ , && (2.25)\\
C_{\bar{A}u\bar{B}r} &\sim O(r^{-3}) \ , && (2.26)\\
C_{\bar{A}r\bar{B}r} &\sim O(r^{-4}) \ , && (2.27)\\
C_{\bar{A}r\bar{B}\vartheta} &\sim O(r^{-3}) \ , && (2.28)\\
C_{\bar{A}\vartheta\bar{B}\vartheta} &\sim O(r^{-1}) \ ; && (2.29)
\end{aligned}$$

and from Eqs. (2.9) and (2.10)

$$k^\vartheta \sim O(r^{-1}) \ , \quad k^\phi = 0 \ . \tag{2.30}$$

Hence we have

$$W_{\bar{A}\bar{B}} = - C_{\bar{A}u\bar{B}u}(k^u)^2 + O(u_p^{-2}) \ . \tag{2.31}$$

These Weyl components are related to the Bondi news, N_{AB} [49]:

$$C_{\bar{A}u\bar{B}u} = -\frac{1}{2r}\frac{d}{du}N_{AB} + O(r^{-2}) \ . \tag{2.32}$$

We have introduced an unbarred basis defined by $e^a{}_A = re^a{}_{\bar{A}}$, with the feature that in this basis boundary quantities such as C_{AB} and N_{AB} are independent of r. Unbarred capital indices will be raised and lowered with the unit two sphere metric, h_{AB}.

Since the expansion of $H(u_p)$ is of order G/r^2, the $\theta\,\varsigma_{\bar{A}\bar{B}}$ term in Eq. (2.22) is always subleading in our analysis. Because the Bondi news and the shear of $H(u_p)$ both vanish in the far future, Eq. (2.22) implies

$$\varsigma_{\bar{A}\bar{B}} = \frac{1}{2r}N_{AB}\cos^2(\vartheta/2) + O(r^{-2}) \ , \tag{2.33}$$

where we have used $d^2u/dw^2 \sim O(r^{-1})$.

On the other hand, the "boundary shear tensor" appearing in Eq. (2.4) was defined in Ref. [25] as

$$\hat{\varsigma}_{ab}(u,\vartheta,\phi) \equiv \frac{1}{\sqrt{8\pi G}}\lim_{r\to\infty} r\,\frac{\varsigma_{ab}(u,r,\vartheta,\phi)}{\cos^2(\vartheta/2)} \ . \tag{2.34}$$

Comparing the previous two equations and using Eq. (2.21), we recognize that the boundary shear is the Bondi news, up to an $O(1)$ rescaling:[3]

$$\hat{\varsigma}_{AB} = \frac{N_{AB}}{\sqrt{32\pi G}} \ . \tag{2.35}$$

The factor of $G^{-1/2}$ ensures that $\hat{\varsigma}^2$ has the dimension of an energy flux.

Returning to the definition of the total asymptotic energy flux, Eq. (2.4), we can now write $\mathcal{T}$ in terms of the Bondi news:

$$\hat{\mathcal{T}} = \hat{T}_{uu} + \frac{1}{32\pi G}N_{AB}N^{AB} \ , \tag{2.36}$$

Note that the definition of the boundary shear $\hat{\varsigma}_{AB}$ was tied to a family of null planes $H(u_p)$ whose orientation picks out a special point on the sphere. Since the Bondi news admits an independent definition that does not require us to pick such a point, it follows that the asymptotic bounds derived in Ref. [25] are independent of the orientation of the $H(u_p)$.

In the remainder of this paper, we will specialize to the case where all outgoing radiation is gravitational. Then $\hat{T}_{uu} = 0$ and $\hat{\mathcal{T}} = N_{AB}N^{AB}/32\pi G$. We see that the square of the Bondi news controls the entropy flux of gravitational radiation.

[3] In the Newman-Penrose formalism, the Bondi news is commonly identified with the u-derivative of the shear of the family of *outgoing* null congruences specified by $u =$ const [75]. Here we relate the news to the shear of *ingoing* null congruences.

2.3 Implications of the Equivalence Principle

In this section, we consider classical aspects of gravitational radiation. We derive consequences of the equivalence principle: the hypothesis that no subset of Minkowski space contains any measurable classical information. Since we use the notion of classical information throughout this and the following sections, we begin with a simple example of such information and its description, in Sec. 2.3.

It is possible to find nonvacuum *quantum* states whose effective stress tensor (analogous to Eq. (2.36)) vanishes in a bounded region. The geometry in this region could be Riemann-flat, yet the region could contain quantum information. Here we only assume the absence of classical information in Minkowski space. In particular we assume that no observable is associated with a coordinate choice in Minkowski space.

The geometry of Minkowski space is trivial, but of course the coordinates are arbitrary. So the matrix of metric components can take many different forms, both generally and in the asymptotic region. Restricting to Bondi coordinates does not fully fix this ambiguity. The equivalence principle implies that any parameters of the Bondi metric that are not unique in Minkowski space must be unobservable, or else that parameter would constitute measurable information. There is no $\hbar$ in the metric of Minkowski space in any Bondi gauge, so the corresponding coordinate information would be classical information.

This includes in particular a parameter C_{AB} (defined below) that has been interpreted [7, 92, 58] as labelling degenerate vacua (Sec. 2.3). Indeed, no observation of this parameter has yet been made, and we are not aware of a proposal for how it could be measured.

A key consequence of the equivalence principle is that the gravitational memory created by Bondi news can be measured only by recording the news. It cannot be measured by probing the vacuum before and after the news (Sec. 2.3). Finally, we note that the equivalence principle does not preclude soft gravitational radiation from carrying information, if "soft" is understood in the physically relevant sense of a small expansion parameter (Sec. 2.3).

These conclusions are in harmony with our findings in Sec. 2.4, where we apply the bounds of Sec. 2.2 to constrain the information content of gravitons and of the vacuum.

Classical Information

A simple example is a classical n-bit message written by Alice and delivered to Bob, say as a sequence of red and blue balls shot across space. Before Bob looks at the balls, he is ignorant of their state. Thus he can describe it as a density operator in a 2^n dimensional Hilbert space, which is diagonal in the $\{\text{red, blue}\}^n$ basis, with equal probability 2^{-n} for each possible message. The Shannon and von Neumann entropies are both

$$-\sum_{i=1}^{2^n} p_i \log p_i = -\mathrm{Tr}\rho \log \rho = n \log 2 \ . \tag{2.37}$$

This is an incoherent superposition, or classical probabilistic ensemble (not to be confused with a coherent quantum superposition of ball sequences).

By looking at the balls, Bob learns Alice's message. Alice cannot send Bob more information than the maximum entropy of the system that carries the message. Since we can express Bob's initial ignorance as a density operator, quantum entropy bounds limit classical communication, as a special case.

Of course, the full quantum Hilbert space is much larger due to the internal degrees of freedom of the balls. And even in the tiny subfactor spanned by $\{\text{red, blue}\}^n$, more general states are possible at the quantum level, which are not product states of the individual balls.

But for classical messages represented by a quantum density operator ρ_i, the ensemble interpretation [77] implies that the full density operator can be written as

$$\rho = \sum_{i=1}^{2^n} \rho_i \ . \tag{2.38}$$

Since the ρ_i are classically distinguishable—and therefore mutually orthogonal—states, there is an irreducible uncertainty in the von Neumann entropy: the entropy cannot be parametrically less than the classical value, $n \log 2$. At the field theory level, this will remain true for the vacuum-subtracted von Neumann entropy: it must be parametrically at least $n \log 2$ (assuming the region contains all balls), since the vacuum entanglement is an ultraviolet quantum property shared by all the classical states.

In this paper we often consider the equivalence principle: the statement that Minkowski space, and any subset of it, contain no classical information. It is worth reflecting on what it would mean if empty Riemann flat space did contain measurable information. In that case it could be used by Alice to communicate a message to Bob.

To be concrete, consider an arbitrarily large patch of flat space (say, the interior of a falling elevator, or a large void in our universe). For it to contain information in an operationally meaningful sense, Alice would have to be capable of "preparing" this region, perhaps by sending a certain sequence of gravitational waves through it. Later, long after those waves have left the region and it is again empty and Riemann-flat, Bob would have to be capable of reading out the message that Alice "left behind", by examining only this patch.

Specifically, if $c(\Omega)$ was observable, then independent observers with access only to the flat space region, would all come to the same conclusion as to which coordinates should be used to label its spacetime points. More precisely, up to corrections subleading in $1/r$, such observers would uniquely identify topological spheres on which the Bondi coordinates u and r must be constant, thus partially fixing the chart. This would indeed be a textbook violation of the equivalence principle.

Empty Space Has No Classical Information

Let us consider the asymptotic metric of an asymptotically flat spacetime, in standard retarded Bondi coordinates (see, e.g., [22, 90, 7, 92, 93, 49]):

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m_B(u,\Omega)}{r}\right) du^2 - 2\,du\,dr \\ &+ r^2\left(h_{AB} + \frac{C_{AB}(u,\Omega)}{r}\right) \times \\ &\quad \times \left(d\vartheta^A + \frac{D_C C^{AC}}{2r^2} du\right)\left(d\vartheta^B + \frac{D_C C^{CB}}{2r^2} du\right) \\ &+ \ \dots \end{aligned} \tag{2.39}$$

where m_B is the Bondi mass aspect, and the ellipses indicate terms subleading in r (consult Appendix A for interpretation and transformation properties of m_B). Here, $C_{AB}(u,\Omega)$ appears as the $1/r$ correction to the round two-sphere metric h_{AB}. It satisfies $h^{AB}C_{AB} = 0$ and $C_{AB} = C_{BA}$. The Bondi news is defined by

$$N_{AB} = \partial_u C_{AB} \ . \tag{2.40}$$

In Minkowski space, the news vanishes. However, the asymptotic metric of Minkowski space can be written in the form of Eq. (2.39), with $m_B \equiv 0$ and any u-independent choice of a tracefree symmetric $C_{AB}(\Omega)$ satisfying

$$C_{AB} = (2D_A D_B - h_{AB} D_C D^C)\, c(\Omega) \tag{2.41}$$

for some function c on the sphere. But of course, the geometry is always the same, no matter how we label its points. There is no curvature of any kind, whatever value we choose for $c(\Omega)$. By the equivalence principle, this implies that c and C_{AB} cannot be measured.

C_{AB} does transform nontrivially under large diffeomorphisms of the asymptotic metric [92, 18, 16, 49]; indeed, this is one way to see that it is non-unique in Minkowski space. Under a BMS supertranslation, $u \to u + f(\Omega)$, one has

$$C_{AB} \to C_{AB} + \left(2D_A D_B - h_{AB} D^C D_C\right) f(\Omega) \tag{2.42}$$

in regions where $N_{AB} = 0$. This corresponds to a well-defined *change* in the shape of a large coordinate sphere at constant u, r. It affects all such spheres equally; for example $C_{AB}(\infty)$ and $C_{AB}(-\infty)$ will change by the same amount under a supertranslation. Of course, this does not imply that C_{AB} is observable. A coordinate sphere is not a physical object but a collection of spacetime points. Its initial shape before the transformation is set by a coordinate choice.

The transformation properties of C_{AB} under supertranslations have been interpreted as an infinite "vacuum degeneracy" of Minkowski space [92, 7, 58]. Each "vacuum" is labeled by the function $c(\Omega)$ in Eq. (2.41). We conclude that the equivalence principle precludes any observable consequences of this degeneracy.[4] (Refs. [8, 42] give an argument that the vacua are indistinguishable starting from different assumptions.)

Gravitational Memory

In nonvacuum spacetimes, C_{AB} need not be constant in u, and differences between C_{AB} at different cuts are observable as "gravitational memory." However, the value of C_{AB} at any one cut (or its zero-mode) must be unobservable in any asymptotically flat spacetime, or else the equivalence principle would be violated in regions where no news arrives. We will now discuss this.

Suppose that some process (a binary inspiral, say) produces gravitational radiation, and that the corresponding Bondi news arrives entirely between the cuts $\hat{\sigma}_1$ and $\hat{\sigma}_2$ of $\mathcal{I}^+$. The integral of the news along the null direction is called the gravitational memory produced by the process,

$$\Delta C_{AB}(\Omega) \equiv \int_{\hat{\sigma}_1}^{\hat{\sigma}_2} du\, N_{AB}(u, \Omega) \tag{2.43}$$

By Eq. (2.43), the production of memory requires nonzero flux of radiation, N_{AB}. Hence memory production occurs only in excited states, not in the vacuum. For example, a graviton wavepacket can produce memory; but then the global state is not the vacuum, but a one-particle state. This qualitative fact continues to hold invariantly in the "soft limit," as the wavepacket is taken to have arbitrarily large wavelength.

What is the physical manifestation of ΔC_{AB}, or equivalently, how can it be measured? In Sec. 2.2, we showed that N_{AB} is proportional to the shear of a planar null congruence $H(u_p)$ near $\mathcal{I}^+$. Hence the gravitational memory is related to the integrated shear, i.e., the resulting strain of the congruence. The displacement vector $\eta^{\bar{A}}$ of two infinitesimally nearby null geodesics will change by

$$\Delta \eta^{\bar{A}} = \Delta C^{A}{}_{B}\, \frac{\eta^{\bar{B}}}{2r} \tag{2.44}$$

between $\hat{\sigma}_1$ and $\hat{\sigma}_2$. This can be measured by setting up (before $\hat{\sigma}_1$) a collection of physical, massless particles propagating along the null geodesics that constitute $H(u_p)$, and observing their transverse location on a screen that they hit after $\hat{\sigma}_2$. ΔC_{AB} can also be measured using an array of timelike detectors distributed over a large sphere. The displacement of any two detectors similarly suffers an overall change given by Eq. (2.44).

In general, the memory captures only a small fraction of the information that arrives in distant regions: the integral of the Bondi news. It would certainly be nice to measure this

[4] Note that the equivalence principle only precludes diffeomorphisms from transforming the classical vacuum into a physically distinct configuration. The equivalence principle does not imply that large diffeomorphisms always act trivially. When acting on an excited state, a supertranslation generically produces a distinct excited state, for example with a different relative timing of the Bondi news arriving at different angles.

component using gravitational wave detectors [40, 1]. Such a measurement would not take infinite time, and it would not be conceptually distinct from any other measurement of the outgoing radiation.

By Eq. (2.40) we can write the gravitational memory, Eq. (2.43), as a difference of the metric quantity C_{AB} evaluated at the two cuts,

$$\Delta C_{AB}(\Omega) = C_{AB}[\hat{\sigma}_2] - C_{AB}[\hat{\sigma}_1] \ . \tag{2.45}$$

If C_{AB} is interpreted as labelling a vacuum, the creation of gravitational memory by news could be described as a "transition" between two such vacua. However, according to the equivalence principle this language is misleading, because C_{AB} cannot be observed at a local cut. A "vacuum" in the above sense is a coordinate label that contains no physical information.

Only the difference ΔC_{AB} is invariant (up to Lorentz transformations [49]) and so can be observed. ΔC_{AB} is nonzero only in global states which are *not* the vacuum, and it is fully determined by the integral of the Bondi news. So the function $C_{AB}(u, \Omega)$ contains no physical information beyond what is already in its u-derivative, the news N_{AB}. The observable memory, ΔC_{AB}, captures a subset of the information in the news.

By the equivalence principle, ΔC_{AB} can only be measured by an observer who has access to the entire region in which news arrives. For example, if physical test particles are introduced into the asymptotic region, and their initial position at $\hat{\sigma}_1$ is recorded, then the memory ΔC_{AB} can be measured at $\hat{\sigma}_2$ by observing the new location of these physical objects. This is an integrated measurement of the Bondi news, with the dynamics of the test masses doing the integration.

Formally, it can be convenient to consider the "zero mode" of the news,

$$C_{AB}(\Omega, \infty) - C_{AB}(\Omega, -\infty) \equiv \int_{-\infty}^{\infty} du \, N_{AB}(\Omega, u) \ . \tag{2.46}$$

This quantity represents the total amount of memory produced in an asymptotically flat spacetime. As written, it is *not* observable, since no experiment began in the infinite past and will end in the infinite future. Fortunately, in any physical process or sequence of processes, the production of news will have a beginning and an end. So one can record the entire memory in a finite-duration experiment, corresponding to a sufficiently large finite range of integration.

To summarize both this and the previous subsection, the value of C_{AB} at any one cut can be changed by a global change of coordinates. By the equivalence principle, C_{AB} cannot be observed and contains no physical information. Therefore, in particular, we cannot measure the gravitational memory, ΔC_{AB} by observing C_{AB} locally at $\hat{\sigma}_1$ and $\hat{\sigma}_2$ and computing the difference. Rather, physical test masses are essential for recording the news and integrating it to obtain ΔC_{AB} between the two cuts. If we forgot to introduce real test masses at $\hat{\sigma}_1$, we cannot look at empty space at $\hat{\sigma}_2$ and learn anything from it.

Soft Gravitons

A soft particle is an excitation of a massless field whose characteristic wavelength, or inverse frequency, is large compared to some dynamical timescale that otherwise characterizes a problem. For example, consider a binary system composed of neutron stars or black holes. They orbit each other with some frequency ω, which varies slowly as they approach, until they eventually merge. The system will emit "hard" gravitational waves with frequency of order ω. The overall duration of the inspiral process is much greater than ω^{-1}; it is characterized by a second time scale $\tau \gg \omega^{-1}$. Or consider a black hole emitting Hawking radiation. The wavelength of the "hard part" of the radiation is of order the black hole radius, $\omega^{-1} \sim O(R)$, which changes slowly. Nonetheless, the overall process takes a much longer time, $\tau \sim O(R^3/G\hbar)$.

Because the emission of "hard" radiation slowly transports gravitating energy from the center to distant regions, the gravitational field will vary not only with characteristic frequency ω, but also over the timescale τ. Therefore, signals with characteristic frequency as low as τ^{-1} are produced in the above processes. Such signals are referred to as "soft". (Often the term "soft graviton" is used, even when the signal is classical.)

This terminology is convenient when we wish to distinguish particles associated with different timescales in a given problem. Useful results can be obtained by expanding in ratios of such timescales [103]. It can also be convenient to idealize soft particles by taking a $\tau \to \infty$ limit, for the purposes of making such expansions sharp. It is worth stressing, however, that infinite-duration experiments are not actually needed to produce and measure a soft particle. (If they were, soft particles would have no physical relevance.) The larger time scale τ is necessarily finite in any physical process.

Moreover, the production of observable radiation comes at a nonzero energy cost. If a soft graviton were added to the vacuum, one would obtain an excited state orthogonal to the vacuum, not a new vacuum. This is a qualitative statement, and independent of τ. Thus, there is no fundamental difference between soft particles and any other form of radiation that arrives in distant regions.

Correspondingly, when we apply the boundary entropy bounds of Sec. 2.2 in Sec. 2.4, all Bondi news can be treated on the same footing. For example, if the interval under consideration in Eq. (2.19) or Eq. (2.16) is large enough to contain a news wavepacket (hard or soft), we will find that this graviton will contribute to the energy side, and generically also to the entropy side of the inequality.

2.4 Entropy Bounds on Gravitational Wave Bursts and the Vacuum

In this section, we compute the upper bounds of Sec. 2.2 in simple asymptotically flat spacetimes: Minkowski space, and a burst of Bondi news that creates gravitational memory.

We show that the upper bounds are consistent with constraints derived in the previous section from the equivalence principle.

Vacuum

Let us apply the bounds of Sec. 2.2 to empty Minkowski space: the Boundary QNEC, Eq. (2.11); the Boundary GSL in integrated and differential form, Eqs. (2.14) and (2.19); and the Boundary QBB, Eq. (2.16). All of these bounds are linear in the boundary stress tensor $\hat{\mathcal{T}}$, i.e., quadratic in the Bondi news. Since $\hat{\mathcal{T}} = 0$ in Minkowski space, the upper bounds all vanish.

The Boundary QBB implies that the vacuum-subtracted entropy is nonpositive in any finite subregion of $\mathcal{I}^+$. The Boundary GSL implies that it is nonpositive for any semi-infinite region above a cut, and independent of the choice of region or deformations of the cut. The Boundary QFC implies (redundantly with the above) that the second derivative under deformations also vanishes.

These upper bounds are consistent with all implications of the equivalence principle described in the previous section: no subset of Minkowski space contains any classical information. Moreover, both the bounds and the equivalence principle are consistent with the simplest possibility for the quantum description of Minkowski space: that the ground state is unique, and that the vacuum-subtracted entropy precisely vanishes on any subregion of $\mathcal{I}^+$.

Classical Bondi News

For simplicity, we will consider a single wave packet of gravitational radiation, of characteristic wavelength λ in the u-direction. The wave packet is roughly centered on $u = 0$ and delocalized on the sphere. The wave packet can be used to send a message to an observer at $\mathcal{I}^+$, for example by encoding it in its polarization, its shape, its direction (the angle Ω at which it arrives), or the time of arrival, within a finite discrete set of $\mathcal{N}$ possible choices.

For concreteness, let us encode the information in the energy of the wavepacket. We take the energy to be of order E for any message, but with a grading into $\mathcal{N}$ different values. A single graviton has energy of order $\hbar/\lambda$. Since we wish to work in the classical regime, the grading must be much coarser than that, so the number of distinct classical states will satisfy

$$\mathcal{N} \ll \frac{E\lambda}{\hbar} , \tag{2.47}$$

We assume that any of the distinct classical signals arrives with equal probability $1/\mathcal{N}$. The classical Shannon entropy is thus $\log \mathcal{N}$.

If we apply the Boundary QBB, Eq. (2.16), or the Boundary GSL, Eq. (2.19), to the region occupied by the wavepacket, we obtain

$$\hat{S}_C \lesssim \frac{E\lambda}{\hbar} \tag{2.48}$$

This is consistent: in our example, the vacuum-subtracted entropy need not be much greater than the Shannon entropy $\log \mathcal{N}$, which is much smaller than $\mathcal{N}$ and hence, by Eq. (2.47), much smaller than the upper bound. Thus, the asymptotic bounds of Sec. 2.2 easily accommodate the classical information contained in the Bondi news.

On the other hand, if we apply the same bounds to a region that fails to overlap with the wavepacket, then the upper bound vanishes:

$$\hat{S}_C \leq 0 \ . \tag{2.49}$$

This is consistent with the absence of classical information in asymptotic regions that do not contain news, as required by the equivalence principle.

In particular, the bounds are consistent with our conclusion in Sec. 2.3 that gravitational memory can only be measured by an observer who has access to the news that creates the memory. In our present example, the news is featureless but for its overall energy. So its integral, the memory, contains the same amount of information as the news, $\log \mathcal{N}$. [We have $\mathcal{T} \sim N_{AB} N^{AB}/G \sim E/\lambda$, so the memory will be of order $\Delta C_{AB} \sim N_{AB}\lambda \sim (GE\lambda)^{1/2}$.] By Eq. (2.49), this information is unavailable to an observer who cannot access the news.

Chapter 3

Asymptotic Charges Cannot Be Measured in Finite Time

3.1 Communication Without Energy?

Alice would like to send Bob a message. Alice lives on a small, massive planet. Bob occupies a Dyson sphere of large radius r_B and negligible mass, which surrounds Alice in an otherwise empty, asymptotically flat spacetime (see Fig. 3.1). It would be simplest for Alice to send Bob a radio signal, or some gravitational waves. Unfortunately, their sleep schedules are out of sync, so that Bob would not be awake when Alice's signal arrives. Instead, they come up with an ingenious protocol, which makes it unnecessary for Bob to intercept any signal from Alice.

Their protocol is as follows. Long ago, before Bob traveled to the Dyson sphere, Alice told Bob the mass M_0 of her planet. She promised not to radiate any of it away until the agreed time when the message is to be sent. That fateful night, she radiates away a certain portion of the mass of her planet. The radiation passes through Bob's sphere while he sleeps, without interacting, and is lost forever.

But when Bob wakes up, he measures the new Bondi mass M of Alice's planet. This can be done at arbitrary distance, by measuring the surface integral that defines the Bondi mass (see Eqs. (3.1) and (3.2) below).

Alice and Bob have agreed on a code, whereby the possible values of M are binned into discrete intervals, and each interval means a particular message. For example, suppose that Alice's planet has initial mass $M_0 = 10^{24}$ kg, and Bob is able to measure the final Bondi mass M to a resolution of 1 kg. Then Alice can choose from among 10^{24} messages. Upon measuring M, Bob gains an amount $\log 10^{24}$ of information, or about 80 bits.

Alice and Bob believe that their scheme will work, given a sufficiently long but fixed, finite retarded time δu for Bob to perform measurements after he wakes up, no matter how big the Dyson sphere is. That is, it should succeed in the limit as $r_B \to \infty$ at fixed retarded time $u \equiv t - r$ and fixed δu (see Fig. 3.1).

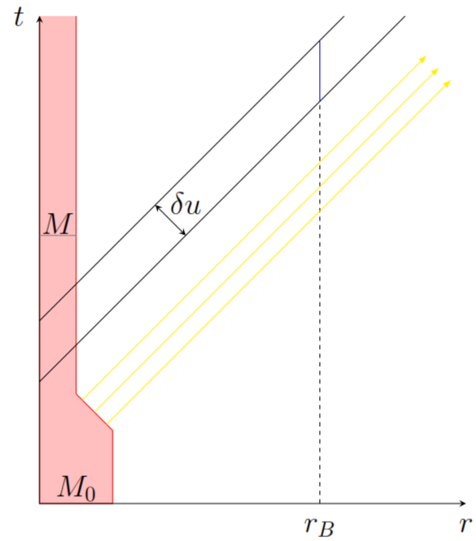


Figure 3.1: If distant observer Bob could measure the Bondi mass of Alice's planet, then Bob could receive information from Alice, without receiving energy. This would contradict recently proven bounds on distant communication channel capacities. In our example, Alice has radiated away some portion of her planet, but Bob does not intercept this radiation (yellow arrows). Instead, Bob later tries to measure how much mass is still left, in some fixed amount of time δu, at arbitrarily large radius r_B. We resolve the contradiction by showing that quantum fluctuations ruin Bob's measurement. The Bondi mass cannot be observed in finite time.

The restriction to fixed u and δu at arbitrarily large r_B is very important to Bob, because he likes to finish all his work before his mid-afternoon nap. It is also important to many theorists, who wish to associate a Bondi mass (and other charges) to a "cut," or cross-section, of future null infinity $\mathscr{I}^+$, which lies at infinite r and is parametrized by u. Of course, no measurement can be performed truly instantaneously, so Bob instead pursues the more modest goal of measuring the Bondi mass in some finite retarded time interval of length δu.

The formal definition of the Bondi mass is associated with a constant-u cut of future null infinity, $\mathscr{I}^+$ (see Fig. 3.2). To make contact with this definition, we consider the limit of a very large Dyson sphere, $r_B \to \infty$, at fixed retarded time u_0 in the metric

$$ds^2 = -\left(1 - \frac{2m_B}{r}\right) du^2 - 2du\,dr + r^2 d\Omega^2 + \dots \tag{3.1}$$

The ellipsis indicates terms subleading in $1/r$ that we will not need. Here m_B is the Bondi mass aspect. Its integral over a 2-sphere cut of $\mathcal{I}^+$ yields the Bondi mass:

$$M = \frac{1}{4\pi} \int_{S^2} d^2\Omega \; m_B \tag{3.2}$$

To claim that an asymptotic observer can measure the Bondi mass in finite time, is to claim that M can be determined by measurements in a distant region $\mathcal{R}$ in Fig. 3.2. Here $\mathcal{R}$ is bounded on the inside by an arbitrarily large radius r_B, and in the past and future by the lightsheets $u = u_0 \pm \frac{\delta u}{2}$.

However, if this protocol succeeded, we would have a paradox. Building on universal entropy bounds [35, 34, 37, 36, 25, 32], it was recently shown that communication from Alice to Bob is constrained by a universal limit on the mutual information that can be achieved [28].

In the limit as $r_B \to \infty$, the amount of information that can be gained by Bob is of order $E\delta u$, where E is the average energy of the signal that is actually received by his detectors. More precisely, the entropy in the detection region is bounded by the modular energy K in the interval δu:

$$K = \int d^2\Omega \int_{u_1(\Omega)}^{u_2(\Omega)} du\, g(u)\, \mathcal{T}(u, \Omega) \ . \tag{3.3}$$

Here Ω is the angle on the sphere at $\mathscr{I}^+$; $\mathcal{T} = \lim r^2 T_{uu}$ is the energy flux arriving on $\mathscr{I}^+$ per unit angle and unit retarded time; and $g(u)$ is a positive definite function. (For a free field, $g(u) = \frac{(u_2-u)(u-u_1)}{u_2-u_1}$.) But K vanishes because $\mathcal{T}$ vanishes: Bob receives no energy at all. He missed the radiation Alice sent earlier, and by the time he measures the mass or charge, there is no radiative flux at all. The entropy is closely related to the Holevo quantity [28], which bounds the mutual information between Alice and Bob. Hence, Bob cannot learn anything from Alice in this protocol.

In light of this contradiction, it is natural to go back and ask where the troublesome bound on communication [28] came from. It was obtained [25, 32] as a limit of the "Quantum Bousso bound," which was proven for free field theories in [37] and for interacting theories in [36]. Ultimately, this entropy bound arose from the conjecture [24, 48] that the entropy in a region is bounded by the cross-sectional area loss along a *lightsheet* traversing the region, measured in Planck units. Here, the lightsheet is a family of parallel light-rays that pass through the asymptotic region. Radiation will focus such light-rays, and the area they span will contract by an amount that remains fixed in Planck units, as the location of the family is taken to infinite distance. The curvature due to the Schwarzschild metric of Alice's planet will also focus the light-rays (through a shear term), but it is easy to check that the resulting area loss goes to zero as the lightsheet is taken off to null infinity.

Thus, Alice and Bob's protocol must fail: it cannot be possible to extract information by measuring a conserved charge in fixed finite time at arbitrarily large distance. In this paper, we will show how it fails. We find that, in the limit as $r_B \to \infty$ at fixed δu, quantum fluctuations dominate and prevent Bob from measuring the conserved charge.[1]

This does not mean, of course, that it is impossible to measure a conserved charge at great distances. It just cannot be done in fixed finite time. As long as the duration of the

[1]Astronomical determinations of mass are performed in the opposite limit, $\delta u \gg r_B$, and so are unconstrained by our analysis. For example, the mass of the Sun can be found by measuring the period of Earth and applying Kepler's Third Law. In such an experiment one has $r_B = 1$ A.U. ≈ 8 min $\ll \delta u \sim 1$ year.

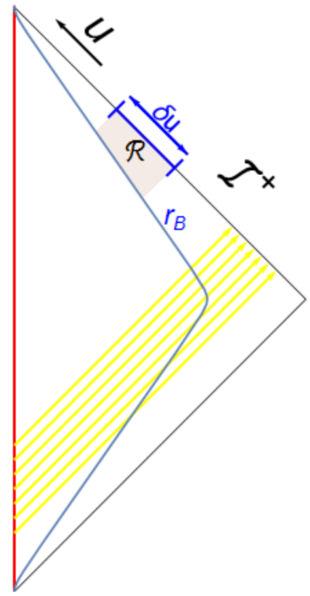


Figure 3.2: Penrose diagram of the process we consider. The red line represents Alice's worldline. The yellow arrows are the radiation emitted by Alice and reaching $\mathscr{I}^+$ without interacting with Bob (blue worldline) whose detectors are only on for a retarded time interval δu.

measurement scales as an appropriate positive power of r, it is possible to determine the charge. But then the measurement cannot be associated with a finite neighborhood of a cut at future null infinity. Rather, the support of any successful measurement must approach (at least) a semi-infinite region of $\mathscr{I}^+$ in the large r limit. Similar comments apply to charges defined at spatial infinity, such as the ADM mass. They are defined by taking $r \to \infty$ at fixed t rather than fixed u. Again the duration of the measurement must scale as a positive power of r to control fluctuations.

Outline In Sec. 3.2 we begin with warm-up problem: we consider charge fluctuations near future null infinity in massless QED. We turn to the gravitational case in Sec. 3.3. An appendix contains details of our calculations.

3.2 Bondi electric charge

In standard QED, the charged particles are massive. Here we consider massless QED, as a closer analogue to the above thought-experiment where Alice uses a massless field (gravitons) to radiate away part of her planet's mass. Translated to the setting of massless QED, the paradox outlined above persists: Alice's planet now starts out with some nonzero charge

Q_0, and Alice reduces this charge to Q by emitting massless charged particles. The charged radiation crosses Bob's sphere while he sleeps, so when he later attempts to determine Q, he does so by measuring the radial electric field E_r integrated over his Dyson sphere, and applying Gauss's law:

$$Q = r_B^2 \oint E_r(\Omega) d^2\Omega \ , \tag{3.4}$$

where Ω is the solid angle on the sphere.

The fluctuation of the electric charge in some region, $\langle Q^2 \rangle$, can be computed by integrating the two-point function of the timelike component of the current density, $\langle j^0(x) j^0(y) \rangle$. Note that Bob does not attempt to measure Q by integration of a charge density over a volume. Bob has access only to an asymptotic region, so naturally he would try to measure Q by integrating the radial electric field over the boundary of the volume. But by Gauss's law, this is the same operator. Here we find it easier to evaluate its fluctuations using the volume form of the operator.

In any CFT, the two-point function is fixed by conformal invariance. In flat space the $U(1)$ current two-point function just takes the form [52],

$$\langle j^0(x) j^0(y) \rangle = \kappa \frac{|\vec{\Delta}|^2 + (\Delta^0)^2}{\Delta^8} \ , \tag{3.5}$$

where $\Delta = x - y$, and the constant κ is theory dependent. For massless Dirac fermions, the current and the propagator are given by [84]

$$\begin{aligned} j^\mu &= \bar{\psi}\gamma^\mu\psi \ , && (3.6) \\ \langle \bar{\psi}(x)\psi(y) \rangle &= -\frac{i}{2\pi^2} \frac{\gamma_\mu (x^\mu - y^\mu)}{(x-y)^4} \ , && (3.7) \end{aligned}$$

which leads to $\kappa_{(\frac{1}{2})} = -\frac{1}{\pi^4}$. For comparison, in massless scalar QED one has[2]

$$\begin{aligned} j^\mu &= i\left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi\right) \ , && (3.8) \\ \langle \phi^*(x)\phi(y) \rangle &= \frac{1}{4\pi^2 (x-y)^2}, && (3.9) \end{aligned}$$

which gives $\kappa_{(0)} = -\frac{1}{4\pi^4}$.

In the 2-point functions (3.7) and (3.9), an $i\epsilon$ prescription must be specified. The choice

$$\Delta^0 \to \Delta^0 - i\epsilon \tag{3.10}$$

[2]This is the leading order result. Scalar QED is not really scale-invariant, due to the nontrivial renormalization group flow of the couplings. Unlike a massless fermion field, ϕ can gain a mass by renormalization. Even if one tunes the field to be massless, there will still be a logarithmic screening of the QED coupling constant as we flow to the IR. However, since we find a power law divergence for $\langle Q^2 \rangle$ at leading order, it does not seem possible that this divergence can be removed by a logarithmic effect. Thus we expect our qualitative conclusions to be the same for massless scalar QED, as for the fermion.

allows for only non-negative energy states in the spectrum. In the complex Δ^0-plane this corresponds to a contour prescription that cuts above both poles in Eq. (3.5). In what follows, this prescription will be implicit.

The total charge inside a spatial region V at the time t_B of Bob's measurement is

$$Q[V] = \int_V d^3x\, j^0(x) \; ; \tag{3.11}$$

but as an operator this would have divergent fluctuations. To obtain a well-defined operator, we smear over a finite time,

$$Q = \int dt\, Q[V(t)]\, w(t) \; . \tag{3.12}$$

The weight function $w(t)$ is normalized so that $\int_{-\infty}^{\infty} w(t)dt = 1$. It should peak in a finite time interval of characteristic size δt, centered on t_B; and it should fall off rapidly outside this interval. Our choice

$$w(t) = \frac{\delta t}{\pi} \frac{1}{(t - t_B)^2 + \delta t^2} \; , \tag{3.13}$$

facilitates the application of contour integration methods. Any other choice with a fast enough fall off should lead to the same qualitative behavior.

For $V(t)$, we must choose the volume enclosed by Bob's Dyson sphere, which is a round ball centered at the origin. Because its radius is much greater than the expected support of the charge (Alice's planet), $\langle Q \rangle$ will not depend on its precise choice. Thus we can allow for a time-dependent radius, for example as

$$r(t) = r_B + \alpha(t - t_B) \; . \tag{3.14}$$

Physically, this corresponds to the freedom to let Bob's Dyson sphere expand or contract during the measurement.[3] This turns out to give Bob more freedom to suppress fluctuations, but nevertheless we will find that they diverge.

We are interested in the limit as Bob's radius goes to infinity along a lightcone, $r_B = t_B + u_B \to \infty$, so that Q becomes the Bondi charge. By an overall time shift, we may set the fixed retarded time of Bob's measurement to zero, $u_B = 0$. We can then fix the retarded time duration of Bob's measurement, as the interval $-\frac{\delta u}{2} < u < \frac{\delta u}{2}$. That is, the weight function (3.13) should have support when Bob's world tube (3.14) lies in this interval, but not outside it. To this end we choose

$$\delta t = \frac{\delta u}{1 - \alpha}. \tag{3.15}$$

[3]One might worry that $r(t)$ is negative for $t < t_B - \frac{r_B}{\alpha}$. However, since this happens only at the tail of the weight function $w(t)$ (Eq. (3.13)), it does not affect our results. For example, the choice $r(t) = r_B(1 - \alpha \tanh(r_B)) + \alpha t \tanh(t)$, which has the same behavior as Eq. (3.14) at large t and is nowhere negative, leads to the same asymptotic behavior.

Note that the proper time duration of Bob's measurement is then given by

$$\delta\tau = \delta u\sqrt{\frac{1+\alpha}{1-\alpha}}. \tag{3.16}$$

Intuitively, we might expect that fluctuations will be more suppressed for greater $\delta\tau$, i.e., for Bob's sphere expanding at great velocity, $\alpha \to 1$. However, as we shall see this is not sufficient to control the fluctuations as $r_B \to \infty$.

To evaluate $\langle Q^2\rangle$, we now write it as

$$\langle Q^2\rangle = \int d^dx \int d^d\Delta\, w(x^0)w(y^0)\theta(\vec{x})\theta(\vec{x}-\vec{\Delta})\langle j^0(0)j^0(\Delta)\rangle, \tag{3.17}$$

where $\theta = 1$ inside the volume V and $\theta = 0$ outside.

Here we summarize how this calculation goes. More details can be found in the Appendix. The integral over $d^3\vec{x}$ yields the volume of the intersection of two balls separated by $|\vec{\Delta}|$. By spherical symmetry, the integral over $d^3\vec{\Delta}$ reduces to a one-dimensional integral which we evaluate. We subsequently perform the dx^0 and $d\Delta^0$ integrations using contour methods. Here one has to be careful to choose a contour that properly avoids branch cuts. This yields an expression for $\langle Q^2\rangle$ as a function of $r_B, \delta t$, and thus via Eq. (3.15), of $r_B, \delta u, \alpha$.

$$\begin{aligned} \langle Q^2\rangle &= -\kappa\left(\pi^2\frac{(1-\alpha)^3 r_B^2}{3(\alpha+1)\delta u^2} + \frac{\pi^2}{6}\log\left(\frac{4(1-\alpha)^3 r_B^2}{(\alpha+1)\delta u^2}\right)\right) \\ &- \frac{\kappa\pi^2}{12(\alpha^2-1)} + O\left(r_B^{-1}\right) \end{aligned} \tag{3.18}$$

We can now take the limit $r_B \to \infty$. For $\alpha = 0$, we find an expected area law divergence. For other choices of α, it is possible to have $\langle Q^2\rangle$ diverge slower than that. To accomplish the goal of making $\langle Q^2\rangle$ grow as slow as possible with r_B, the optimal choice of α satisfies

$$1-\alpha^{\text{opt}} \propto \sqrt{\frac{\delta u}{r_B}}, \tag{3.19}$$

No choice of α can make $\langle Q^2\rangle$ diverge slower than that, and in particular, no choice of α can make the charge fluctuations finite when $r_B \to \infty$. For the optimal choice above, the divergence goes as the fourth-root of the area,

$$\langle Q^2\rangle^{\text{opt}} \sim \sqrt{\frac{r_B}{\delta u}}\ . \tag{3.20}$$

The results above are for four dimensional Minkowski space, but the same analysis can be performed in any dimension (though we have only been able to get analytic results in even dimensions). Here we quote the results in two[4] and six dimensions:

[4]Since QED is confining in 2D, one cannot give the 2D result the same interpretation as in higher dimensions.

$$\langle Q^2\rangle_{D=2} \;\propto\; \log\left(\frac{\left(\delta u^2+(1-\alpha)^4 {r_B}^2\right)^2}{\left(1-\alpha^2\right)^2 \delta u^4}\right) \tag{3.21}$$

$$\langle Q^2\rangle_{D=6} \;\propto\; \frac{(1-\alpha)^6 {r_B}^4}{(\alpha+1)^2\delta u^4}+O\left({r_B}^2\right) \tag{3.22}$$

We see that for constant α, we always get an area law $\langle Q^2\rangle_D \sim \left(\frac{r_B}{\delta u}\right)^{D-2}$. The optimal choice of α is always given by Eq. (3.19) for any D; this yields

$$\langle Q^2\rangle_D^{\mathrm{opt}} \sim r_B^{(D-2)/4} \sim \delta\tau^{D-2}\ . \tag{3.23}$$

This divergence thwarts Bob's plans of measuring the charge and thus prevents him from receiving Alice's message. Since no information is transmitted, the apparent paradox described in the previous section is resolved.

3.3 Bondi mass

In the previous section we showed that, due to quantum fluctuations, the Bondi electric charge cannot be measured in a finite interval of $\mathscr{I}^+$. Here we repeat this analysis, but for the Bondi mass. For concreteness, we consider a massless scalar field non-minimally coupled to gravity. However, since the two point function of T_{00} is completely fixed (up to a multiplicative factor) in any scale-invariant theory with a stress-tensor, our conclusions apply equally well to spinors, gauge fields, and interacting fixed points.

The action and stress-energy tensor for a non-minimally coupled scalar are given by

$$S=-\frac{1}{2}\int d^4\sqrt{-g}\left(D_\mu\phi D^\mu\phi+\xi R\phi^2\right), \tag{3.24}$$

and

$$\begin{aligned} T_{\alpha\beta} &= (1-2\xi)D_\alpha\phi D_\beta\phi+\left(2\xi-\frac{1}{2}\right)D_\mu\phi D^\mu\phi g_{\alpha\beta} \\ &+ 2\xi g_{\alpha\beta}\phi D^2\phi-2\xi\phi D_\alpha D_\beta\phi. \end{aligned} \tag{3.25}$$

Using this stress-energy tensor and $\langle\phi(0)\phi(\varDelta)\rangle=\frac{1}{\varDelta^2}$, we get

$$\langle T_{00}(x)T_{00}(y)\rangle = 8\left(30\xi^2-10\xi+1\right)\frac{3\vec{\varDelta}^4+10\varDelta_0^2\vec{\varDelta}^2+3\varDelta_0^4}{\left(\varDelta_0^2-\vec{\varDelta}^2\right)^6}. \tag{3.26}$$

Using the same smearing as in the previous section, we can now calculate the fluctuations of the energy,

$$\langle M^2\rangle=\int d^4x\int d^4\varDelta\, w(x^0)w(y^0)\theta(\vec{x})\theta(\vec{x}-\vec{\varDelta})\langle T_{00}(x)T_{00}(y)\rangle, \tag{3.27}$$

by performing the same integrals as in the QED case, the details of which are relegated to the Appendix.

As in the $U(1)$ case, we choose to evaluate the operator and its fluctuations as a volume integral, not a surface integral. This is now more subtle, because strictly the Bondi mass is defined *only* as a surface integral over a family of topological 2-spheres $\{S_\alpha\}$ that approach a cut S of null infinity [100]:

$$M = -\lim_{S_\alpha \to S} \frac{1}{8\pi} \int_{S_\alpha} \varepsilon_{abcd} \nabla^c \zeta^d \tag{3.28}$$

where ζ^a is an asymptotic time translation Killing vector field. Here we work in a perturbative limit, where backreaction in the bulk is small. Then an approximate Gauss law still holds, and the Bondi mass can also computed as a volume integral

$$M = \int_{\tilde{\Sigma}} d^3x \; T_{00} \tag{3.29}$$

over the portion $\tilde{\Sigma}$ of a Cauchy surface Σ enclosed by S. Moreover, we can reach arbitrarily large M even in the perturbative regime, by considering matter of low density spread over a large region. Hence we expect that our result for the fluctuations of M will be general.

We find

$$\begin{aligned} \langle M^2 \rangle &= 8\left(30\xi^2 - 10\xi + 1\right)\pi^2 \left(\alpha^2 \delta u^2 + 4(1-\alpha)^4 {r_B}^2\right)^3 \\ &\times \left((1-\alpha)^4\left(3\alpha^2+1\right){r_B}^2 - \left(\alpha^2 - 5\right)\frac{\delta u^2}{4}\right) \\ &\times \left(15(1-\alpha)(\alpha+1)^3 \delta u^4 \left(\delta u^2 + 4(1-\alpha)^4 {r_B}^2\right)^3\right)^{-1} \end{aligned} \tag{3.30}$$

For $\alpha = 0$ this gives

$$\langle M^2 \rangle = 8\left(30\xi^2 - 10\xi + 1\right) \frac{16\pi^2 r_B^6 \left(5\delta u^2 + 4r_B^2\right)}{15\delta u^4 \left(\delta u^2 + 4r_B^2\right)^3}. \tag{3.31}$$

Once again, it is possible to tame this divergence by a better choice of α. The optimal value remains $\alpha^{\text{opt}} \propto 1 - \left(\frac{r_B}{\delta u}\right)^{-1/2}$, which gives

$$\langle M^2 \rangle^{\text{opt}} = \frac{\left(30\xi^2 - 10\xi + 1\right) 2^{5/2}\pi^2}{30\delta u^{5/2}} \sqrt{r_B} + O\left(\frac{1}{r_B^{1/2}}\right). \tag{3.32}$$

We therefore see that the Bondi energy also has unbounded fluctuations as we approach finite intervals of null infinity.

3.4 Discussion

We argued that entropy bounds preclude gauge charges from being well-defined quantum observables on cuts or finite intervals of $\mathscr{I}^+$. We confirmed this by showing that unbounded fluctuations preclude a measurement of the electric charge or the Bondi mass, in finite time at arbitrarily large radius.[5]

It is important to emphasize the quantum nature of these results. Both M and Q are good classical observables near a cut of $\mathscr{I}^+$. This follows directly from Eq. (3.4), and from the analogous surface integral for the Bondi mass, Eq. (3.28). Both expressions are gauge-invariant and require no data extrinsic to the near-cut region $\mathcal{R}$ for their evaluation. This constrasts with certain other quantities appearing in the Bondi metric expansion, Eq. (3.1), which are prohibited by the equivalence principle from being observable already at the classical level (see [32, 23], also appendix A).

Let us try to gain some intuition for the divergence of $\langle Q^2 \rangle$ and $\langle M^2 \rangle$ that we found. To understand the physical origin of the fluctuations, suppose, for simplicity, that Bob remains at fixed radius throughout his measurement, so that $\alpha = 0$ and $\delta u = \delta t = \delta\tau$. Consider Q as a surface integral over E_r, rather than a volume integral. An observation restricted to a finite time interval leads to approximately thermal quantum noise of characteristic energy $1/\delta\tau$. This noise arises in the region causally accessible to the observer; here, this would be a shell of width δt around the sphere r_B. Since $r_B \gg \delta\tau$, there will be a large number $N \sim r_B^2/\delta\tau^2$ of "cells" just inside and outside of Bob's sphere. Each cell contains $O(1)$ quanta of any massless field the detectors couple to, which includes the charges. This contributes to E_r an additional field strength of order $1/\delta\tau^2$ and random sign. The contribution to Q from one cell, in Eq. (3.4), is thus of order ± 1. The fluctuations in different cells are uncorrelated, so the total fluctuation of Q is given by $\langle Q^2 \rangle^{1/2} \sim \sqrt{N} \sim r_B/\delta\tau$. This agrees with Eq. (3.18) for this special case, $\alpha = 0$.[6]

Note that neither infrared nor ultraviolet physics alone can explain the divergent fluctuations of Q and M. Rather, they arise from a combination of both. The fixed duration δu of Bob's measurement sets a characteristic "ultraviolet" energy scale for the fluctuations. The infrared effect comes from taking the limit as $r_B \to \infty$, which creates an ever larger region over which those fluctuations can contribute.

Our work lends some insight on the structure of operator algebras of gauge theories and gravity when quantizing at $\mathscr{I}^+$. We emphasize that the paradox noted in Section 3.1 would arise for any quantity associated to a subset of $\mathscr{I}^+$ that is not tied to energy flux arriving in that subset. For example, the BMS group at $\mathscr{I}^+$ yields an infinite set of supertranslation charges [17], which essentially correspond to the Bondi mass aspect (whose integral yields

[5] The study of fluctuation of electric charge (in finite regions) dates back to the early days of QED (see e.g. [79] and [21]).

[6] It would be nice to extend this heuristic argument to the optimal case, when Bob is expanding outward during the measurement according to Eq. (3.19). But using Eq. (3.16), the above argument would appear to imply $\langle Q^2 \rangle \sim r_B^2/\delta\tau^2 \sim (r_B/\delta u)^{3/2}$, in conflict with Eq. (3.20).

the Bondi mass) [7, 92, 93, 91]. We thus find that these supertranslation charges are not observable in a neighborhood of any cut of $\mathscr{I}^+$ in the quantum theory[7].

The absence of such observables also has potential significance for understanding the holographic principle. There has been considerable interest in trying to construct a holographic theory dual to asymptotically flat spacetimes (see [39, 68, 81] for recent examples). By analogy to AdS/CFT, one expects that such a putative holographic dual should be defined on the conformal boundary of the spacetime, and that limits of bulk observables that are defined as they approach $\mathscr{I}^+$ should correspond to local operators in the putative boundary theory. Since we have shown that conserved charges are not in fact well-defined operators on any finite portion of $\mathscr{I}^+$, we expect that no such operators should exist in a dual boundary theory either.

[7]We established that a certain operator $\hat{O}$ does not belong to the algebra of observables by showing that $\langle \hat{O}^2 \rangle = \infty$. This is not a perfect criterion, since there are contrived examples of observables in quantum mechanics with $\langle \hat{O}^2 \rangle = \infty$ but well-defined spectrum. However, we do expect all *reasonable* operators to have finite fluctuations.

Chapter 4

The Boundary of the Future

4.1 Theorem

In this paper, we prove the following theorem establishing necessary and sufficient conditions for a point to be on the boundary of the future of a surface in spacetime. (An analogous theorem holds for the past of K.)

Theorem 1. *Let (M, g) be a smooth,*[1] *globally hyperbolic spacetime and let K be a smooth codimension-two submanifold of M that is compact and acausal. Then a point $b \in M$ is on the boundary of the future of K if and only if all of the following statements hold:*

(i) b lies on a future-directed null geodesic γ that intersects K orthogonally.

(ii) γ has no points conjugate to K strictly before b.

(iii) γ does not intersect any other null geodesic orthogonal to K strictly between K and b.

Theorem 1 enumerates the conditions under which a light ray, launched normally from a surface, can exit the boundary of the future of that surface and enter its chronological future. In essence, this happens only when the light ray either hits another null geodesic launched orthogonally from the surface or when the light ray encounters a caustic, in a sense that will be made precise in terms of special conditions on the deviation vectors for a family of infinitesimally-separated geodesics. These two possibilities for the fate of the light ray are illustrated in Fig. 4.1.

The theorem is useful for characterizing the causal structure induced by spatial surfaces. In particular, if K splits a Cauchy surface into two parts, then Theorem 1 implies that the four orthogonal null congruences fully characterize the associated split of the spacetime into four portions: the future and past of K and the domains of dependence of each of the two spatial sides (see Fig. 4.2). This is of particular interest when K is a holographic screen [26].

[1] Nowhere in the proof will more than two derivatives be needed, so the assumption of smoothness for M and K can be relaxed everywhere in this paper to C^2.

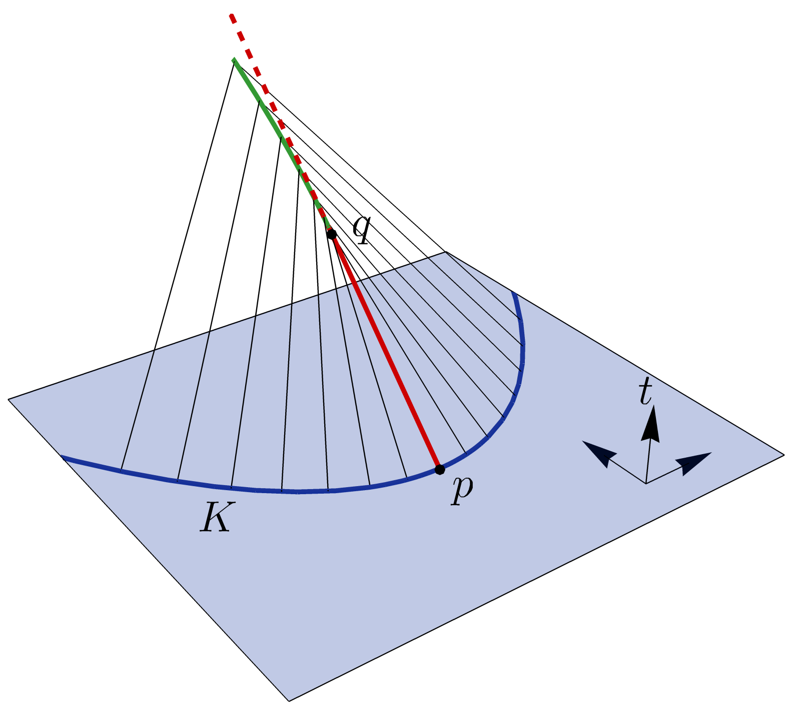


Figure 4.1: Possibilities for how a null geodesic orthogonal to a surface can exit the boundary of its future. In this example, a parabolic surface K (blue line) lies in a particular spatial slice. A future-directed null geodesic (red line) is launched orthogonally from p. At q, it encounters a caustic, entering the interior of the future of K (red dashed line). The point q is conjugate to K. Other null geodesics orthogonal to K (black lines) encounter nonlocal intersections with other such geodesics along the green line, where they exit the boundary of the future of K.

Then some of the orthogonal congruences form *light sheets* [24] such that the entropy of matter on a light sheet is bounded by the area of K. This relation makes precise the notion that the universe is like a "hologram" [64, 94, 46] and should be described as such in a quantum gravity theory. Such holographic theories have indeed been identified for a special class of spacetimes [72].

Specifically, Theorem 1 plays a role in the recent proof of a novel area theorem for holographic screens [30, 31], where it was assumed without proof. It also enters the analogous derivation of a related Generalized Second Law in cosmology [29] from the Quantum Focusing Conjecture [37].

Although our motivation lies in applications to General Relativity and quantum gravity, we stress that the theorem itself is purely a statement about Lorentzian geometry. It does not assume Einstein's equations and so in particular does not assume any conditions on the stress tensor of matter.

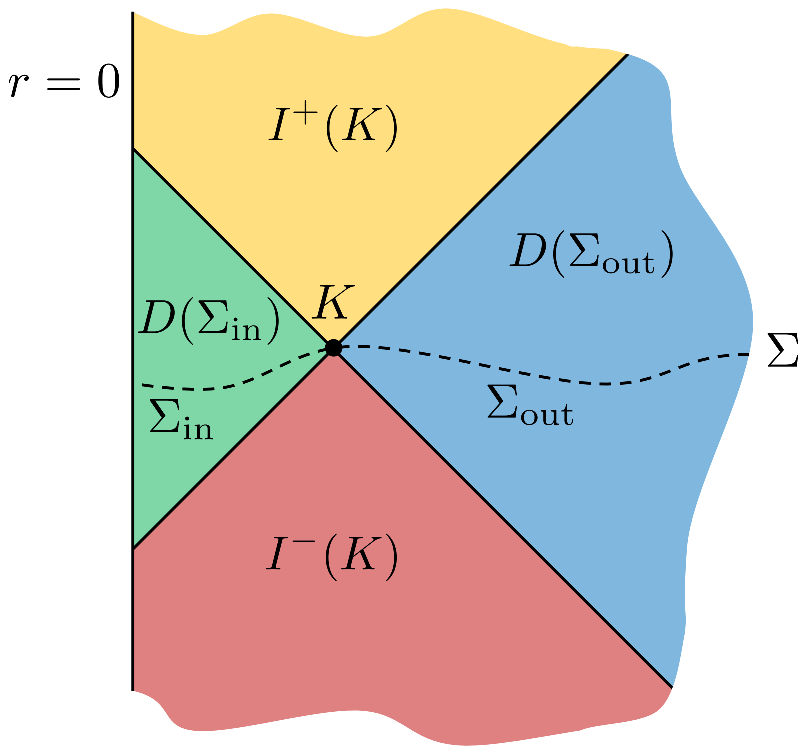


Figure 4.2: In this generic Penrose diagram, the codimension-two surface K (black dot) splits a Cauchy surface Σ (dashed line) into two parts Σ_{in}, Σ_{out}. This induces a splitting of the spacetime M into four parts: the past and future of K (red, yellow) and the domains of dependence of Σ_{in} and Σ_{out} (green, blue) [31]. Theorem 1 guarantees that this splitting is fully characterized by the four orthogonal null congruences originating on K (black diagonal lines).

Related Work. Parts of the "only if" direction of the theorem are a standard textbook result [100], except for (*iii*), which we easily establish. The "if" direction is nontrivial and takes up the bulk of our proof.

Ref. [19] considers the *cut locus*, i.e., the set of all *cut points* associated with geodesics starting at some point $p \in M$. Given a geodesic γ originating at p, a future null cut point, in particular, can be defined in terms of the Lorentzian distance function or equivalently as the final point on γ that is in the boundary of the future of p. As shown in Theorem 5.3 of Ref. [19], if q is the future null cut point on γ of p, then either q corresponds to the first future conjugate point of p along γ, or another null geodesic from p intersects γ at q, or both. Our theorem can be viewed as an analogous result for geodesics orthogonal to codimension-two surfaces and a generalization of our theorem implies the result of Ref. [19]

as a special case. The codimension-two surfaces treated by our theorem are of significant physical interest due to the important role of holographic screens in the study of quantum gravity (see, e.g., Ref. [43] for very recent results on the coarse-grained black hole entropy). We encountered nontrivial differences in proving the theorem for surfaces. Moreover our condition (*ii*) places stronger constraints on the associated deviation vector, as we discuss in Sec. 4.2.[2]

The previously known parts of the "only if" direction of Theorem 1 were originally established in the context of proving singularity theorems [83, 54]. It would be interesting to see whether Theorem 1 can be used to derive new or stronger results on the formation or the cosmic censorship of spacetime singularities.

Generalizations. As we are only concerned with the causal structure, the metric can be freely conformally rescaled. Thus, a version of Theorem 1 still holds for noncompact K, as long as it is compact in the conformal completion of the spacetime, i.e., in a Penrose diagram. A situation in which this may be of interest is for surfaces anchored to the boundary of anti-de Sitter space.

Furthermore, the theorem can be generalized to surfaces of codimension other than two, but in that case we can say less about the type of conjugate point that orthogonal null geodesics may encounter. We will discuss this further in Sec. 4.3.

Notation. Throughout, we use standard notation for causal structure. A causal curve is one for which the tangent vector is always timelike or null. The causal (respectively, chronological) future of a set S in our spacetime M, denoted by $J^+(S)$ (respectively, $I^+(S)$) is the set of all $q \in M$ such that there exists $p \in S$ for which there is a future-directed causal (respectively, timelike) curve in M from p to q. For the past ($I^-(S)$, $J^-(S)$, etc.), similar definitions apply. We will denote the boundary of a set S by $\dot{S}$. Standard results [100] include that $I^\pm(S)$ is open and that $\dot{J}^\pm(S) = \dot{I}^\pm(S)$. We will call a set S *acausal* if there do not exist distinct $p, q \in S$ for which there is a causal path in M from p to q. A spacetime is said to be *globally hyperbolic* if it contains no closed causal curves and if $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. Equivalently [51], M has the topology of $\Sigma \times \mathbb{R}$ for some *Cauchy surface* Σ; that is, Σ is a surface for which, for all $p \in M$, every inextendible timelike curve through p intersects Σ exactly once.

Outline. In Sec. 4.2, we review the notion of a conjugate point and establish some useful lemmas. In Sec. 4.3, we prove Theorem 1.

[2] After this paper first appeared, we were made aware of Refs. [71, 69], which also generalize the results of Ref. [19] to codimension-two surfaces. Our work goes further in that we more strongly constrain the type of conjugacy to be that of Def. 17. This is crucial for making contact with the notion of points "conjugate to a surface" used in the physics literature, e.g., in Ref. [100].

4.2 Conjugate Points to a Surface

Exponential Map

Let (M, g) be a smooth, globally hyperbolic spacetime of dimension $n > 2$. Thus, M is a manifold with metric g of signature $(-, +, \dots, +)$. (As already noted, we will be concerned only with the causal structure of M, so g need only be known up to conformal transformations.)

For $p \in M$, let T_pM be the tangent vector space at p and let $TM \equiv \bigcup_{p\in M}\{p\} \times T_pM$ be the tangent bundle of M. TM has a natural topology that makes it a manifold of dimension $2n$. In the open subsets associated with charts of M, TM is diffeomorphic to open subsets of $\mathbb{R}^{2n}$, corresponding to n coordinates for the location of $p \in M$ and n components of a tangent vector $v \in T_pM$. The tangent space of TM at (p, v) is

$$T_{p,v}TM = T_pM \times T_vT_pM. \tag{4.1}$$

For every $(p, v) \in TM$, there is a unique inextendible geodesic,

$$c_{p,v} : (a, b) \to M, \;\; s \mapsto c_{p,v}(s), \tag{4.2}$$

where $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$, with affine parameter s and tangent vector $v \in T_pM$ given by the pushforward of d/ds by $c_{p,v}$ at the point $p = c_{p,v}(0) \in M$. It is convenient to include the degenerate curves obtained with $v = 0$.

Definition 2. *The* exponential map *is defined by:*[3]

$$\exp \;:\; TM \to M, \;\; (p, v) \mapsto c_{p,v}(1). \tag{4.3}$$

Restrictions of exp to submanifolds of TM are frequently of interest. To study the congruence of geodesics emanating from a given point, one may restrict to $\exp_p : T_pM \to M\,,\;\; v \mapsto c_{p,v}(1)$. Moreover, one can define the differential of $\exp_p$, $\exp_{p*} : T_vT_pM \to T_{c_{p,v}(1)}M$, which describes how $\exp_p v$ varies due to small changes in v. See Fig. 4.3 for an illustration of the exponential map and its differential. In this paper, we will consider a different restriction suited to the study of the geodesics orthogonal to a given spatial surface; we will define the differential in more detail for this restriction below.

Let $K \subset M$ be a smooth submanifold. We consider the normal bundle

$$NK \equiv \bigcup_{p\in K} \{p\} \times T_pK^{\perp},$$

where $T_pK^{\perp}$ is the two-dimensional tangent vector space perpendicular to K at p. The normal bundle has the structure of an n-dimensional manifold. Its tangent space at $(p, v) \in NK$ is

$$T_{p,v}NK = T_pK \times T_vT_pK^{\perp}. \tag{4.4}$$

[3] If the spacetime is not geodesically complete, the exponential map can only be defined on the subset of TM consisting of the (p, v) such that $c_{p,v}$ can be extended to $\lambda = 1$. This restriction will be left implicit in this paper.

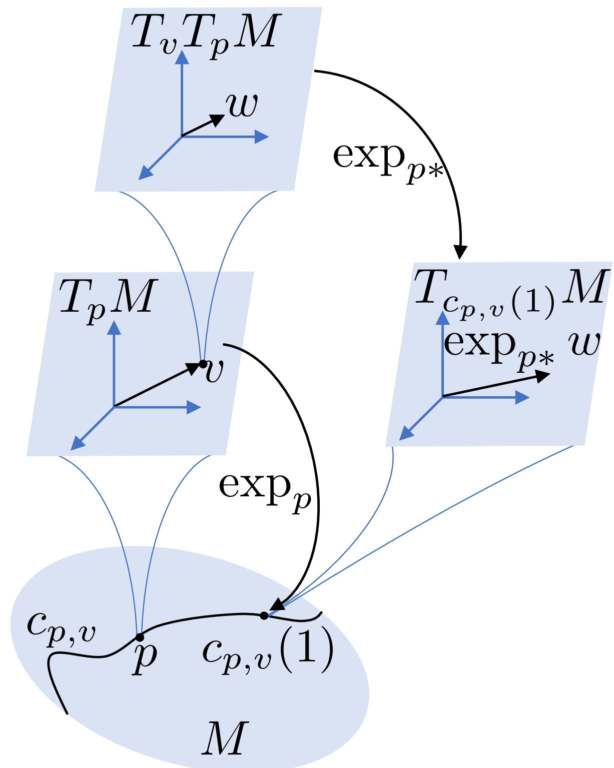


Figure 4.3: An illustration of the exponential map exp, which takes a vector in TM to a point in M, and the Jacobian of the exponential map, which takes a vector in the tangent space TTM of TM to a vector in TM.

Here, T_pK is the tangent space of p in the manifold K; that is, T_pK is the subspace of T_pM normal to $T_pK^\perp$. Note that T_pK is of the same dimension as K.

Definition 3. *The* surface-orthogonal exponential map

$$\exp_K \;:\; NK \to M\,,\;\; (p, v) \mapsto c_{p,v}(1) \tag{4.5}$$

is the restriction of exp *to* NK.

Definition 4. *The* Jacobian *or* differential *of the exponential map is given by*

$$\exp_{K*} \;:\; T_{p,v}NK \to TM\,,\;\; w \mapsto \exp_{K*} w. \tag{4.6}$$

It is a linear map between vectors that captures the response of $\exp_K$ *to small variations in its argument. It is defined by requiring that* $(\exp_{K*} w)(f) = w(f \circ \exp_K)$ *for any function* $f : M \to \mathbb{R}$*. Note* $\exp_{K*} w$ *is the* pushforward *of* w *by* $\exp_K$*. If* x^α *are coordinates in an open neighborhood of* $(p, v) \in NK$ *and* y^β *are coordinates in an open neighborhood* *of* $\exp_K(p,v) \in M$ *and we write the vectors in coordinate form,* $w = \sum w^\alpha(\partial/\partial x^\alpha)$ *and* $\exp_{K*} w = \sum \hat{w}^\beta(\partial/\partial y^\beta)$*, then the components are related by the* Jacobian matrix,

$$\hat{w}^\beta = \sum_\alpha \frac{\partial y^\beta}{\partial x^\alpha} w^\alpha. \tag{4.7}$$

See Fig. 4.4 for an illustration of $\exp_K$, $\exp_{K*}$, and the various tangent spaces used in this paper.

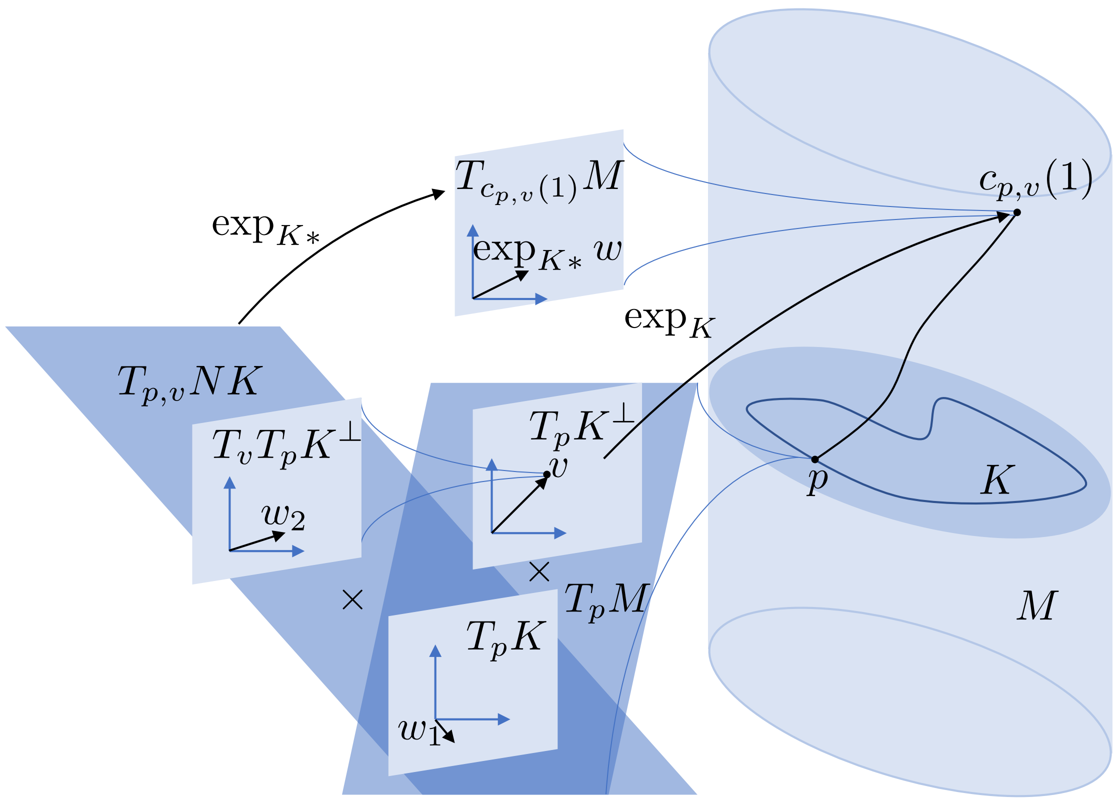


Figure 4.4: An illustration of the surface-orthogonal exponential map $\exp_K$ evaluated at $p \in K$, which takes a vector in $T_pK^\perp$ to a point $c_{p,v}(1)$ in M. Here, as in text, the tangent space at p, T_pM, is broken up as a product $T_pK^\perp \times T_pK$. Also shown is the Jacobian $\exp_{K*}$ at $v \in T_pK^\perp$, which takes a vector $w = (w_1, w_2) \in T_{p,v}NK = T_pK \times T_vT_pK^\perp$ to a vector in $T_{c_{p,v}(1)}M$.

Definition 5. *A Jacobian is an* isomorphism *if it is invertible, i.e., if it has no eigenvectors with eigenvalue zero.*

Since (M,g) and K are smooth, $\exp_K$ is smooth. The inverse function theorem [87] thus implies the following.

Lemma 6. *If the Jacobian* $\exp_{K*}$ *at* $(p,v) \in NK$ *is an isomorphism, then* $\exp_K$ *is a diffeomorphism of an open neighborhood of* (p,v) *onto an open neighborhood of* $\exp_K(p,v) \in M$*.*

Definition 7. *The exponential map* $\exp_K$ *is called* singular *at* $(p,v) \in NK$ *if* $\exp_{K*}$ *is not an isomorphism. Then* (p,v) *is called a* conjugate point *in* NK*.*

Jacobi Fields

It is instructive to relate the above definition of *conjugate point* to an equivalent definition in terms of Jacobi fields.

Definition 8. *Let* Q *be an open set in* $\mathbb{R}^2$ *and let* $f : Q \to M$*,* $(r,s) \mapsto f(r,s)$ *be a smooth map. If the curves of constant* r *and varying* s*,* $\gamma_r : Q \to M, s \mapsto f(r,s)$*, are geodesics in* M*, then* f *is called a* one-parameter family (or congruence) of geodesics*.*

Definition 9. *Let* ∂_s *denote the partial derivative with respect to* s*. It follows from the above definition that the pushforward* $S \equiv f_*(\partial_s) \in TM$ *is tangent to any geodesic* γ_r*. Similarly,* $R \equiv f_*(\partial_r) \in TM$ *is tangent to any curve* $\mu_s : Q \to M, r \mapsto f(r,s)$ *at fixed* s*. For general families of curves,* R *represents the deviation vector field of the congruence. In the special case of a geodesic congruence,* R *restricted to any* γ_r *is called a* Jacobi field *on* γ_r*.*

Remark 10. *The Jacobi field* R *satisfies the geodesic deviation equation on* Q*,*

$$D_S^2 R = \mathcal{R}(S,R)S, \tag{4.8}$$

where $\mathcal{R}(A,B) \equiv [D_A, D_B] - D_{[A,B]}$ *is the curvature tensor [61, 100] and* $D_V = V^\mu \nabla_\mu$ *is the covariant derivative, defined with respect to the Levi-Civita connection, along a vector* V*.*

The exponential map can be used to generate a one-parameter family of geodesics and its derivative $\exp_*$ generates the associated Jacobi fields. We first recall the more familiar case of geodesics through a point p, generated by $\exp_p$, as follows.

Remark 11. *Let* $\hat{R}, \hat{S} \in T_pM$ *and let* $\tilde{R}$ *and* $\tilde{S}$ *be the naturally associated constant vector fields in* TT_pM*.*[4] *Then* $f(r,s) = \exp_p[s(\hat{S} + r\hat{R})]$ *is smooth and defines a one-parameter family of geodesics. Its tangent vector field is* $S = \exp_{p*}\big|_{s(\hat{S}+r\hat{R})} (\tilde{S} + r\tilde{R})$ *and its deviation or Jacobi field is* $R = \exp_{p*}\big|_{s(\hat{S}+r\hat{R})} s\tilde{R}$*.*[5] *It is clear from this construction that* $\exp_p$ *is singular (i.e.,* $\exp_{p*}$ *fails to be an isomorphism) at* $s(\hat{S} + r\hat{R})$ *if and only if there exists a nontrivial Jacobi field of the geodesic* γ_r *that vanishes at* $f(r,s)$ *and* $f(r,0)$*. This establishes the equivalence of two common definitions of conjugacy to a point* p*.*

[4] Concretely, one can first choose a neighborhood U of p diffeomorphic to $\mathbb{R}^n$, which exists since M is a manifold, and then choose a map $\phi : U \to T_pM$ such that the pushforward ϕ_* is the identity map from T_pM to T_vT_pM for some v; then $\tilde{R}$ and $\tilde{S}$ can be defined as $\tilde{R} = \phi_*\hat{R}$ and $\tilde{S} = \phi_*\hat{S}$ for $v = \hat{R}$ or $\hat{S}$, respectively.

[5] The subscript is the point where the Jacobian map is evaluated. The vector the Jacobian acts on appears to its right.

Remark 12. *A conjugate point in a geodesic congruence with tangent vector k^μ corresponds to a* caustic*, which is a point at which the* expansion $\theta = \nabla_\mu k^\mu$ *goes to* $-\infty$.

We turn to the case relevant to this paper: a one-parameter family of geodesics orthogonal to a smooth, compact, acausal, codimension-two submanifold K. (For example, K could be a topological sphere at an instant of time.) Subject to this restriction, the map f and vector fields R and S are defined as before, with T_pM replaced by $T_pK^\perp$. One can choose the parameters (r,s) such that $f(r,0) \in K$ and $f(0,0) = p$. The map $\nu : r \mapsto (f(r,0), S|_{(r,0)})$ is a smooth curve in NK with tangent vector $\bar{R} \in TNK$. From this curve, the one-parameter family can be recovered as

$$f(r,s) = \exp_{f(r,0)} sS|_{(r,0)} = \exp_K(f(r,0), sS|_{(r,0)}). \tag{4.9}$$

Remark 13. *We will be interested in the Jacobi field $R \equiv f_*\partial_r$ only along one geodesic, say γ at $r = 0$. By Eq.* (4.8) *this depends only on the initial data S and $\bar{R}$ at p. Thus $R|_{(0,s)}$ will be the same for any curve ν with tangent vector $\bar{R}$ at $(p, S|_{(0,0)}) \in NK$. Conversely, one can extend any given $\bar{R}$ at $(p, S|_{(0,0)}) \in NK$ to a (non-unique) one-parameter family of geodesics by picking such a curve ν. We now take advantage of this freedom in order to find an explicit expression for the Jacobi field in terms of* $\exp_{K*}$.

By Eq. (4.4), one can uniquely decompose $\bar{R} = (\check{R}, \tilde{R})$, with $\check{R} \in T_pK$ and $\tilde{R} \in T_ST_pK^\perp$. Let π be the defining projection of the fiber bundle, $\pi : NK \to K$. Then $\mu \equiv \pi(\nu)$ is a curve on K with tangent vector $\check{R}$ at p. Let $f(r,0) = \mu(r)$.

Further, let $S|_{(r,0)} \in T_{f(r,0)}K^\perp$ be defined by K-normal parallel transport[6] of the vector $S|_{(0,0)} + r\hat{R} \in T_pK^\perp$ along μ from p to $\mu(r)$. Here $\hat{R} \in T_pK^\perp$ is the vector naturally associated with $\tilde{R} \in T_ST_pK^\perp$. Similarly, we define $\tilde{S} \in T_ST_pK^\perp$ to be the vector naturally associated with $S|_{(0,0)}$.

Lemma 14. *With the above choices and definitions, Eq.* (4.9) *yields a suitable one-parameter family of geodesics. The corresponding Jacobi field and tangent vector along γ can be written as:*

$$R|_{(0,s)} \equiv f_*\partial_r|_{(0,s)} = \exp_{K*}|_{(p,sS|_{(0,0)})}(\check{R}, s\tilde{R}) \tag{4.10}$$

and

$$S|_{(0,s)} \equiv f_*\partial_s|_{(0,s)} = \exp_{K*}|_{(p,sS|_{(0,0)})}(0, \tilde{S}), \tag{4.11}$$

respectively.

See App. E for a proof of Lemma 14 via a direct calculation.

We note that $\check{R}$ and $\tilde{R}$ encode the initial value and derivative, respectively, of R, in accordance with the initial value problem set up in Remark 13. From Eq. (4.10), we obtain a criterion for conjugacy equivalent to that of Def. 7:

[6]Given a vector $v \in T_pK^\perp$, normal parallel transport defines a vector field $v(r)$ along μ normal to K such that the normal component of its covariant derivative along μ vanishes, $D_r^\perp v(r) = 0$. Given $\mu(r)$ and the initial vector in $T_pK^\perp$, $v(r)$ is unique by Lemma 4.40 of Ref. [78].

Remark 15. *In the above notation, the map* $\exp_K$ *is singular at* $(p, sS|_{(0,0)}) \in NK$ *if and only if the geodesic* γ *possesses a nontrivial Jacobi field that vanishes at* $\exp_K(p, sS|_{(0,0)})$ *and is tangent to* K *at* p.

Specifically, our interest lies in null geodesics orthogonal to K. We now show that their conjugate points satisfy an additional criterion on the associated eigenvector of $\exp_{K*}$.

Lemma 16. *Let* γ *be a geodesic orthogonal to* K *at* p, *with conjugate point* $(p, sS|_{(0,0)}) \in NK$. *By Def. 7 there exists a nonzero vector* $\bar{R} \in T_{p,sS|_{(0,0)}}NK$ *such that* $\exp_{K*}|_{(p,sS|_{(0,0)})}\bar{R} = 0$. *If* γ *is null, i.e., if* $\|S|_{(0,0)}\| = 0$, *then the projection of* $\bar{R}$ *onto* T_pK *is nonvanishing:* $\check{R} \neq 0$.

Proof. By Eqs. (4.10) and (4.11), the Jacobi field $R|_{(0,s)}$ is orthogonal to γ at two points: at p (by construction) and (trivially) at the assumed conjugate point. By Lemma 8.7 of Ref. [78], this implies that $R_{(0,s)} \perp S|_{(0,s)}$ for all s. Again using Eqs. (4.10) and (4.11), along with linearity of $\exp_{K*}$, this implies that $\tilde{R} \perp \tilde{S}$ and thus $\exp_{K*}|_{(p,sS|_{(0,0)})}(0, s\tilde{R}) \perp S$.

Prior to the conjugate point, the map $\exp_{K*}$ is a linear isomorphism; hence it maps the (1+1)-dimensional subspace $T_S T_p K^\perp \ni \tilde{R}$ of $T_{p,S}NK$ into a (1+1)-dimensional subspace $\exp_{K*} T_S T_p K^\perp$ of $T_{f(0,1)}M$. This subspace contains both the null tangent vector $S|_{(0,s)}$ and the component $\exp_{K*}|_{(p,sS|_{(0,0)})}(0, s\tilde{R})$ of the Jacobi field R, which is itself a Jacobi field since our choice of initial data $\bar{R}$ was arbitrary. In a (1+1)-dimensional space, the only vectors orthogonal to a null vector S are proportional to S. The general solution to Eq. (4.8) for a Jacobi field proportional to the tangent vector S is $(\alpha + \beta s)S|_{(0,s)}$. Therefore $\exp_{K*}|_{(p,sS|_{(0,0)})}(0, s\tilde{R})$ must have this form for some real constants α, β. At $s = 0$, $\exp_{K*}|_{(p,sS|_{(0,0)})}(0, s\tilde{R})$ vanishes trivially, so $\alpha = 0$.

Now, suppose $\check{R} = 0$, so $R|_{(0,s)}$ is just $\beta s S|_{(0,s)}$. Since our Jacobi field is nontrivial and S does not vanish, we must have $\beta \neq 0$. Thus, $R|_{(s,0)}$ vanishes only at p and hence cannot vanish at $\exp_K(p, sS|_{(0,0)})$. This contradiction implies that $\check{R} \neq 0$. □

We now define a refinement of the notion of a conjugate point.

Definition 17. *Let* $\gamma(s)$ *be a geodesic orthogonal to* K *at* p, *with* $\gamma(0) = p$ *and with conjugate point* (p, v). *Then there exists a nontrivial Jacobi field* $R(s) \in TM$ *that vanishes at* $q = \exp_K(p, v)$ *and is tangent to* K *at* p. *We say that* q *is conjugate to (the surface)* K *if* R *is nonvanishing at* p.

Remark 18. *By Lemma 16,* $\check{R} \neq 0$, *so the Jacobi field associated with* $\bar{R}$ *as defined in Eq.* (4.10) *does not vanish at* p *and hence, if* $(p, sS|_{(0,0)}) \in NK$ *is a conjugate point, then the point* $\exp_K(p, sS|_{(0,0)})$ *is conjugate to* K *for* γ *null.*

Moreover, we can similarly define the notion of a point conjugate to another point.

Definition 19. *Given a nontrivial Jacobi field* R *for a segment* γ *of a geodesic such that* R *vanishes at* p *and* q, *we say that* q *is conjugate to (the point)* p.

See Fig. 4.5 for an illustration of the two types of conjugate points defined in Defs. 17 and 19.

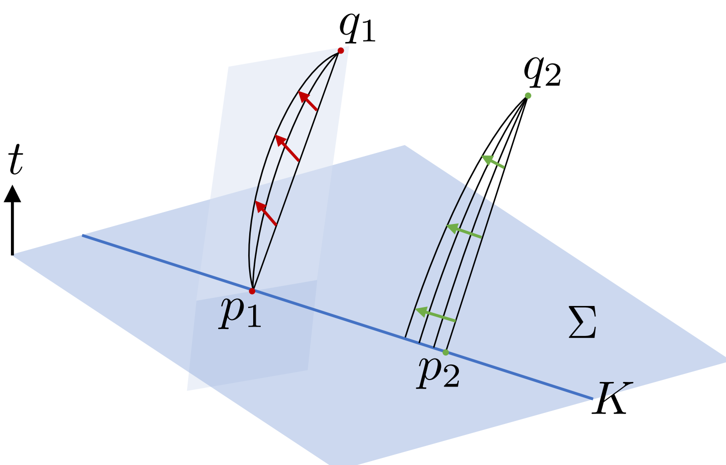


Figure 4.5: The two types of conjugate points defined in Defs. 17 and 19. The point q_1 is conjugate to the *point* p_1, with the Jacobi field illustrated by the red arrows. The point q_2 is conjugate to the *surface* K (blue line), at the point p_2, with the Jacobi field illustrated by the green arrows. Geodesics orthogonal to K are shown in black. If a general conjugate point lies along an orthogonal null geodesic, then by Lemma 16 there exists a Jacobi field such that the conjugate point is of the *surface* type. Hence, this type of conjugacy appears in Theorem 1.

4.3 Proof of the Theorem

We now prove Theorem 1.

Proof. For the "only if" direction, we may assume that $b \in \dot{I}^+(K)$. Then conclusions $(i), (ii)$ are already established explicitly elsewhere in the literature (e.g., Theorem 9.3.11 of Ref. [100] and Theorem 7.27 of Ref. [82]; see also Lemma VII of Ref. [83], as well as Ref. [54]).

Conclusion (iii) follows by contradiction: let γ' be a distinct null geodesic orthogonal to K that intersects γ at some point q strictly between b and K. By acausality of K, $\gamma' \cap K$ is a single point, p', which is distinct from q. Hence, K can be connected to b by a causal curve that is not an unbroken null geodesic, namely, by following γ' from p' to q and γ from q to b. By Proposition 4.5.10 in Ref. [55], this implies that some $r \in K$ can be joined to b by a timelike curve, in contradiction with $b \in \dot{I}^+(K)$. Hence, no such γ' can exist.

The "if" direction of the theorem states that if $(i), (ii), (iii)$ hold, then $b \in \dot{I}^+(K)$. We will prove the following equivalent statement: If $b \notin \dot{I}^+(K)$ satisfies (i), then b will fail to satisfy (ii) or (iii).

Let the geodesic $\gamma(s)$ guaranteed by (i) be parametrized so that $\gamma(0) = p \equiv \gamma \cap K$ and $\gamma(1) = b$. By (i), $b \in J^+(K)$, the causal future of K. By assumption, $b \notin \dot{I}^+(K) = \dot{J}^+(K)$, so it follows that $b \in I^+(K)$, the chronological future of K. Since $p \in \dot{I}^+(K)$, there exists an s_* between 0 and 1 where γ leaves the boundary of the future:

$$s_* \equiv \sup \gamma^{-1}(\gamma([0,1]) \cap \dot{I}^+(K)). \tag{4.12}$$

The point where γ leaves $\dot{I}^+(K)$, $q \equiv \gamma(s_*)$, lies in $\dot{I}^+(K)$.[7] Thus $s_* < 1$. Moreover, $s_* > 0$ by the obvious generalization of Proposition 4.5.1 in Ref. [55] and achronality of K. We conclude that

$$p \in \dot{I}^-(q) \cap K, \quad q \neq b, \text{ and } q \neq p. \tag{4.13}$$

Recall that $q = \gamma(s_*)$ is the future-most point on γ that is not in $I^+(K)$. Let s_n be a strictly decreasing sequence of real numbers that converges to s_*. That is, $s_n > s_*$ and, for n sufficiently large, the points $q_n \equiv \gamma(s_n)$ exist and lie in $I^+(K)$. Now, since K is acausal and M is globally hyperbolic, there exists a Cauchy surface $\Sigma \supset K$. Given $p_1, p_2 \in M$, define $C(p_1, p_2)$ to be the set of all causal curves from p_1 to p_2. Since by Corollary 6.6 of Ref. [82] $C(\Sigma, q_n)$ is compact, it is closed and bounded. Thus, $C(K, q_n) \subset C(\Sigma, q_n)$ is bounded. Consider a sequence of curves μ_m from K to q_n. By Lemma 6.2.1 of Ref. [55], the limit curve μ of $\{\mu_m\}$ is causal; since K is compact and thus contains its limit points, μ runs from K to q_n, so $\mu \in C(K, q_n)$. Hence, $C(K, q_n)$ is closed and therefore compact. Since the proper time is an upper semicontinuous function on $C(\Sigma, q_n)$, it attains its maximum over a compact domain, so we conclude in analogy with Theorem 9.4.5 of Ref. [100] that there exists a timelike geodesic γ_n that maximizes the proper time from K to q_n. By Theorem 9.4.3 of Ref. [100], γ_n is orthogonal to K.

By construction, the point q is a convergence point (and hence a limit point) of the sequence $\{\gamma_n\}$. By the time-reverse of Lemma 6.2.1 of Ref. [55], there exists, through q, a causal limit curve γ' of the sequence $\{\gamma_n\}$. This curve must intersect K because all γ_n intersect K and K is compact. Since γ' passes through $q \in \dot{I}^+(K)$, it must not be smoothly deformable to a timelike curve since $I^+(K)$ is open. Thus, by Theorem 9.3.10 of Ref. [100], γ' must be a null geodesic orthogonal to K, so if $\gamma' \neq \gamma$, condition (iii) fails to hold. See Fig. 4.6 for an illustration.

The only alternative is that γ is the only limit curve of the sequence $\{\gamma_n\}$. In this case, $\{\gamma_n\}$ contains a subsequence whose convergence curve is γ. From now on, let $\{\gamma_n\}$ denote this subsequence. Orthogonality to K of the γ_n implies that we can write

$$q_n = \exp_K(p_n, v_n), \tag{4.14}$$

where $p_n = \gamma_n \cap K$, for some vector $v_n \in T_{p_n}K^\perp$ tangent to γ_n. But since $q_n \in \gamma$, we can also write

$$q_n = \exp_K(p, k_n), \tag{4.15}$$

[7] This follows because $\dot{I}^+(K)$ is closed and hence its intersection with a closed segment of γ is closed. Therefore, the argument of the supremum is a closed interval and the supremum is its upper endpoint.

where k_n is tangent to γ. Thus, every q_n has a non-unique pre-image.

By the above construction, the sequences $\{(p, k_n)\}$ and $\{(p_n, v_n)\}$ in NK each have (p, v) as their limit point, where $q = \exp_K(p, v)$. Hence there exists no open neighborhood $\mathcal{O}$ of (p, v) such that $\exp_K$ is a diffeomorphism of $\mathcal{O}$ onto an open neighborhood of q. By Lemma 6, it follows that $\exp_K$ is singular at (p, v), i.e., (p, v) is a conjugate point. By Lemma 16 and Remark 18, q is conjugate to K. Thus, condition (ii) fails to hold; again, see Fig. 4.6. ■

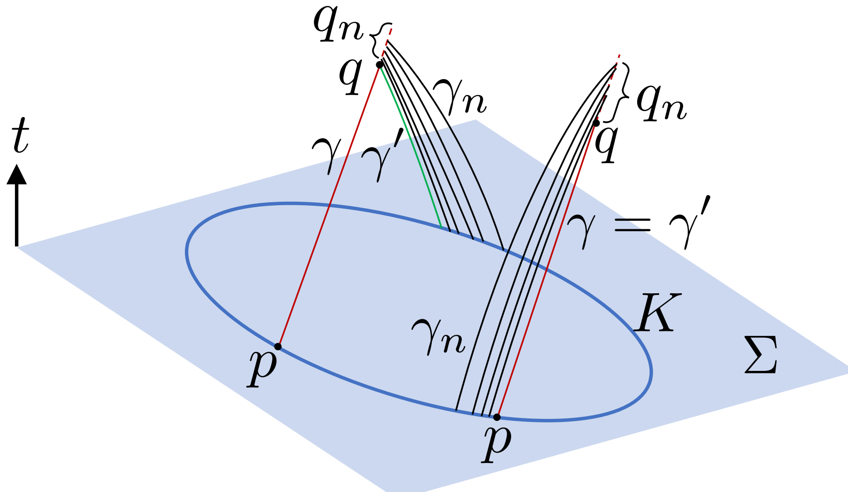


Figure 4.6: Possibilities in the proof. The sequence of timelike geodesics γ_n (black) connects K with a sequence of points $q_n \in I^+(K)$ on the orthogonal null geodesic γ (red) that joins $p \in K$ with q, after which γ leaves $\dot{I}^+(K)$ (red dashed). In the case on the left, γ' (green) is distinct from γ, so condition (iii) fails. In the case on the right, $\gamma' = \gamma$, which we prove corresponds to a failure of condition (ii).

Remark 20. *The fact that K had codimension two was only important in the last step in the proof of Theorem 1, i.e., going from knowing that (p, v) is a conjugate point to showing that q is conjugate to the surface K. For K a compact, acausal submanifold that is not of codimension two, the steps in the proof of Theorem 1 still establish that (p, v) is a conjugate point in the sense of Def. 7. Moreover, that the corresponding Jacobi field is orthogonal to S remains true without the codimension-two assumption (see the proof of Lemma 16) and the one-parameter family of geodesics is orthogonal to K (because it was defined via normal parallel transport). As a result, the Jacobi field defines a deviation of γ in terms of only orthogonal null geodesics (as proven in, e.g., Corollary 10.40 of Ref. [78]), but in general that will not mean that q is conjugate to the surface K in the sense of Def. 17. Specifically, the Jacobi field is not necessarily nonvanishing at K if K has codimension greater than two.*

Chapter 5

Holographic Inequalities and Entanglement of Purification

5.1 Introduction

There has been much recent interest in the interplay between the fields of quantum information and quantum gravity. One central point of interest is on the discussions of notions of entanglement measures in the context of the AdS/CFT correspondence [72, 104]. In particular, the holographic formula relating entanglement entropy to bulk area of a boundary homologous minimal surface by Ryu and Takayanagi [88, 89] (later extended to extremal surfaces in the covariant case by [66]) has spurred a great deal of interest, from its ability to constrain what sets of states can be dual to classical bulk gravity theories [15, 56] to its role as motivation for the idea that the gravity theory is emergent from the entanglement properties of the boundary field theory [98, 73, 45, 95].

Even more recently work has been done [76, 96] conjecturing that a holographic object, the entanglement wedge cross section E_W separating two regions, is dual to the information theoretic concept of the entanglement of purification E_p. This conjecture, which we refer to as the $E_W = E_p$ conjecture, was made on the basis that E_W, a holographic object, obeys the same set of inequalities that E_p is known to obey. This would be a compelling correspondence, as it is not known how to calculate E_p for generic quantum states, whereas E_W is an often finite geometric quantity that is simply calculable.

In this work, we will study and generalize the relationship between E_W, E_p, and the holographic entanglement entropy inequalities in three ways: first, we investigate whether E_W can nontrivially bound combinations of entanglement entropies that appear in the holographic entropy inequalities; second, we check whether E_p provably provides the same type of bounds to these objects; lastly, we ask whether one can extend the $E_W = E_p$ conjecture to suboptimal purifications and cuts of the entanglement wedge. We will find that the answers to all three of these questions appear to be affirmative, thus providing more evidence for the $E_W = E_p$ conjecture of [76, 96].

5.2 Review of known results

Basic properties of E_W and E_p

Let us define both the entanglement wedge cross section E_W, and the entanglement of purification E_p. First, we define holographic states to be quantum states of the boundary conformal field theory that are dual to a well defined classical bulk gravitational theory in AdS/CFT. For a holographic state, the **entanglement wedge cross-section** is defined for any two regions of time reversal symmetric slices (though the generalization to the fully covariant case exists in [96]) as

$$E_W(A:B) = \min\{\text{Area}(\Gamma); \Gamma \subset r_{AB} \text{ splits } A \text{ and } B\} \tag{5.1}$$

where r_{AB} is the entanglement wedge[1] [60] of $AB = A \cup B$ (see Figure 5.1). In words, E_W is the minimal area of a surface Γ that splits r_{AB} into two regions, one of which is bounded by A but not B, and other by B but not A. If s_A, s_B and s_{AB} denote Ryu-Takayanagi (RT) surfaces, then we want Γ to split $r_{AB} = r_{AB}^{(A)} \sqcup r_{AB}^{(B)}$ (here $\sqcup$ denotes disjoint union) and $s_{AB} = s_{AB}^{(A)} \sqcup s_{AB}^{(B)}$ with $\partial r_{AB}^{(A)} = A \cup s_{AB}^{(A)} \cup \Gamma$.[2] In this work, we refer interchangeably to the surface and the area thereof as the entanglement wedge cross-section, but the meaning should be clear from context.

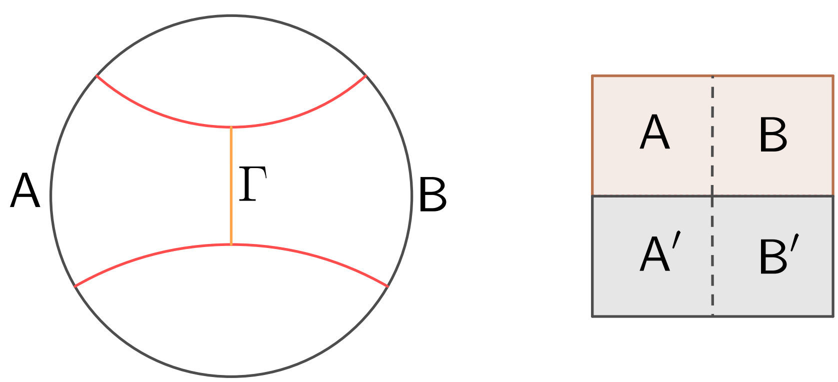


Figure 5.1: To the left, Γ is the minimal surface the separates the entanglement wedge cross-section of AB. Its area is $E_W[A:B]$. To the right, A' and B' purify AB. For a choice of A' and B' over all such purifying systems that minimizes the entanglement across the dashed partition we have $E_p(A:B) = S(AA')$.

Now, consider an arbitrary bipartite quantum system AB. The **entanglement of purification** $E_p(A:B)$ is defined by

[1] Technically, r_{AB} is the restriction of the entanglement wedge to the time symmetric slice under consideration. Since we will only be concerned with this time symmetric situation, and all objects considered live on this slice, we leave this restriction implicit throughout.

[2] It is also worth noting that E_W (and its to-be-developed generalization) is finite if the regions being split are nonadjacent in the boundary theory.

$$E_p(A:B) = \min\{S(AA'); AA'BB'\text{pure}\} \tag{5.2}$$

where S is the Von Neumann entropy (see Figure 5.1). Note that because the overall state is pure this is symmetric under $A \leftrightarrow B$.

Both E_W and E_p are known to satisfy the following inequalities:

$$\min(S_A, S_B) \geq E(A:B) \geq \frac{1}{2} I(A:B) \tag{5.3}$$

$$E(A:BC) \geq E(A:B) \tag{5.4}$$

$$E(AB:C) \geq \frac{1}{2}\left(I(A:C) + I(B:C)\right), \tag{5.5}$$

where E here can stand for either E_p or E_W, and $I(A:B) \equiv S(A) + S(B) - S(AB)$ is the mutual information between A and B. We refer the reader to [96] for clear proofs of these inequalities in the context of E_W, and [97, 9] for the same for E_p. These coinciding bounds for E_p and E_W is what motivated the conjecture of [76, 96] that E_W is the holographic dual of E_p.

Entanglement Entropy Inequalities

Before we study potential new inequalities for E_p and E_W, let's list some known inequalities for entanglement entropy that will prove useful in the upcoming discussion. For holographic proofs of these inequalities we refer the reader to [15, 56, 59]. All tripartite quantum states satisfy strong subadditivity (SSA):

$$I(A:B|C) \equiv S(BC) + S(AC) - S(ABC) - S(C) \geq 0, \tag{5.6}$$

where $I(A:B|C)$ is the conditional mutual information. When $C = \emptyset$, this reduces to subadditivity, or positivity of the mutual information. Moreover, all *holographic* states satisfy monogamy of mutual information (MMI) [56]:

$$I(A:BC) \geq I(A:B) + I(A:C) \Leftrightarrow I(A:B:C) \equiv I(A:BC) - I(A:B) - I(A:C) \geq 0, \tag{5.7}$$

where $I(A:B:C)$ is the tripartite information, which is symmetric under permutations of its arguments.

It is worth stressing that not all quantum states satisfy MMI. For example, the GHZ state defined by $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\right)$ does not do so for $n \geq 4$.

Recently, several further holographic entanglement entropy inequalities were proven [15]. Among them, there's an infinite family of cyclic inequalities given by

$$C_k(A_1, \ldots, A_n) \equiv \sum_{i=1}^{n} S(A_i|A_{i+1}\ldots A_{i+k}) - S(A_1\ldots A_n) \geq 0, \tag{5.8}$$

where $n = 2k+1$, the indices are interpreted mod n, $S(A|B) = S(AB) - S(B)$ is the conditional entropy, and C_k is what we call the k-cyclic information (or just cyclic information). For $k = 1$, Eq. (5.8) gives MMI, but for $k > 1$ it gives a family of new and independent inequalities.

5.3 Bounding holographic entanglement entropy with E_W

Generalized E_W

In order to bound holographic entanglement entropy with E_W, we first slightly generalize the notion of entanglement wedge cross-section $E_W(A : B)$ to allow for $A \cap B \neq \emptyset$. A generalization of $E_p(A : B)$ for this case will be given in section 5.4.

The **generalized entanglement wedge cross-section E_W^G** is defined as

$$E_W^G(A : B) = \min\{\text{Area}(\mathit{\Gamma}); \mathit{\Gamma} \subset r_{AB} - r_{A\cap B} \text{ splits } A\backslash B \text{ and } B\backslash A\}, \tag{5.9}$$

so now the surface $\mathit{\Gamma}$ separates $A\backslash B$ from $B\backslash A$ in the region defined by the entanglement wedge of AB with the entanglement wedge of $A \cap B$ removed[3]. Note that if $A \cap B = \emptyset$ then $E_W^G(A : B) = E_W(A : B)$.

We also define a convenient form of mutual information, $I^G(A : B) = I(A\backslash B, B\backslash A) = S(A\backslash B) + S(A\backslash B) - S(A\backslash B \cup B\backslash A)$. Similiarly, if the intersection between A and B is trivial this reduces to $I(A : B)$.

E_W^G obeys known E_W inequalities

In this section, we show that this generalized entanglement wedge cross-section obeys the known inequalities for E_W in Eqs. (5.3–5.5). In section 5.4 we will show that a suitably generalized E_p also obeys these inequalities. Thus there is as much evidence for the generalized conjecture $E_W^G = E_p^G$ as there is for the original $E_W = E_p$ conjecture.

The upper bound in Eq. (5.3) follows from

$$\begin{aligned} E_W^G(A : B) &\leq E_W(A : B\backslash A) \leq \min(S(A), S(B\backslash A))\text{and} \\ E_W^G(A : B) &\leq E_W(A\backslash B : B) \leq \min(S(A\backslash B), S(B)), \end{aligned} \tag{5.10}$$

where the first inequalities above follow from the fact that for $E_W^G(A : B)$ is the minimum area curve $\mathit{\Gamma}$ that separates $A\backslash B$ from $B\backslash A$ in $r_{AB} - r_{A\cap B}$, and so it can be no longer than optimal curve separating these same regions in r_{AB}.

[3]For an illustration of what this generalization means geometrically, see Fig. 5.2, in which $E_W^G(AC : BC) = \text{Area}(\mathit{\Gamma}))$

The lower bound follows from $r_{AB\backslash(A\cap B)} \subset (r_{AB} - r_{A\cap B})$, which is a consequence of entanglement wedge nesting (EWN) [102, 5] and implies

$$E_W^G(A:B) \geq E_W(A\backslash B : B\backslash A) \geq \frac{1}{2} I^G(A:B). \tag{5.11}$$

It follows from entanglement wedge nesting that if $A \cap C = \emptyset$, then

$$E_W^G(A:BC) \geq E_W^G(A:B). \tag{5.12}$$

Finally,

$$E_W^G(A:BC) \geq \frac{1}{2}\left(I^G(A:B) + I(A\backslash B : C)\right) \tag{5.13}$$

follows from Eq. (5.11), MMI, and the disjointedness of A and C.

Upper bounding holographic conditional mutual information

One can ask the question of whether or not holography in general, and E_W^G in particular, provides an upper bound to the conditional mutual information. We note that this question was first answered in the affirmative by [50] using bit threads, but it is instructive to treat it again here.

The holographic bound for $I(A:B|C)$ in [50] reads:

$$I(A:B|C) \leq 2E_W^G(AC:BC). \tag{5.14}$$

Note that in the case where $C = \emptyset$, this reduces to $I(A:B) \leq 2E_W(A:B)$.

This upper bound can also be proven using exclusion/inclusion [59] or equivalently graph contraction [15], with the main new technique used being that the cutting and regluing procedure is no longer constrained to only boundary anchored minimal surfaces, but potentially includes bulk-anchored minimal surfaces such as the entanglement wedge cross-section as well. See Figure 5.2.

Let's now follow [59] in putting into equations what Figure 5.2 shows. Let s_X denote the RT surface of some boundary region X, and r_X denote its entanglement wedge so that the boundary of r_X is $\partial r_X = X \cup s_X$.

Let A, B, and C be disjoint regions, and let $R = r_{ABC}\backslash r_C$. By EWN, $\partial R = s_{abc} \cup s_c$. Let Γ be the surface that satisfies the minimization in $E_W[AC:BC]$. Then, it splits R into two disjoint regions $R^{(A)}$ and $R^{(B)}$ such that

$$\begin{aligned}
\partial R^{(A)} &= A + \Gamma + s_{ABC}^{(A)} + s_C^{(A)} \\
\partial R^{(B)} &= B + \Gamma + s_{ABC}^{(B)} + s_C^{(B)},
\end{aligned} \tag{5.15}$$

where $s_C = s_C^{(A)} \sqcup s_C^B$ and $s_{ABC} = s_{ABC}^{(A)} \sqcup s_{ABC}^B$. Then,

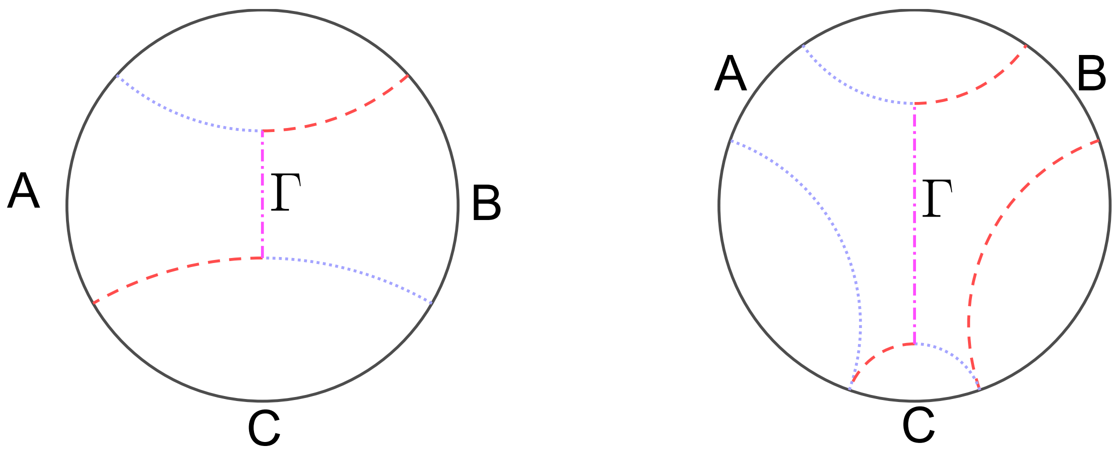


Figure 5.2: Graphical proof of the upper bound on the conditional mutual information for both the case in which the regions A, B, and C are contiguous and the case in which they are disconnected. It is clear from the diagrams,and Ryu-Takayanagi, that the area of the dotted surfaces plus the area of the dash-dotted surface is greater than or equal $S(AC)$ and that the area of the dashed surfaces plus the area of the dash-dotted surface is greater than or equal $S(BC)$. Adding these two inequalities gives us the desired bound.

$$\partial(r_C \cup R^{(A)}) = (A \cup C) \cup (\mathit{\Gamma} \cup s_C^{(B)} \cup s_{ABC}^{(A)}) \tag{5.16}$$

$$\partial(r_C \cup R^{(B)}) = (A \cup B) \cup (\mathit{\Gamma} \cup s_C^{(A)} \cup s_{ABC}^{(B)}) \tag{5.17}$$

Then, by RT, Eq. (5.16) implies that the area of $(\mathit{\Gamma} \cup s_C^{(B)} \cup s_{ABC}^{(A)})$ is greater than or equal to S_{AC}, and Eq. (5.16) implies that the area of $(\mathit{\Gamma} \cup s_C^{(A)} \cup s_{ABC}^{(B)})$ is greater than or equal to S_{BC}. Adding these two inequality, and applying RT again, we get the desired inequality

$$2E_W[AC : BC] \geq S_{AC} + S_{BC} - S_{ABC} - S_C = I(A : B|C) \geq 0, \tag{5.18}$$

where we used the positivity of the conditional mutual information.

Upper bounding holographic tripartite information

We can also use E_W^G to upper bound holographic tripartite information:

$$I(A : B : C) \leq E_W^G(AC : BC) + E_W^G(AB : BC) + E_W^G(CB : AC). \tag{5.19}$$

We could have pursued an inclusion-exclusion style proof for this, but amusingly one does not have to; this follows from Eq. (5.14). Adding three instances of Eq. (5.14), we get:

$$\begin{aligned}
&E_W^G(AC:BC)+E_W^G(AB:CB)+E_W^G(BA:CA)\\
&\geq \frac{1}{2}\left(I(A:B|C)+I(A:C|B)+I(B:C|A)\right)\\
&= S_{AB}+S_{AC}+S_{BC}-\frac{3}{2}S_{ABC}-\frac{1}{2}\left(S_A+S_B+S_C\right)\\
&\geq S_{AB}+S_{AC}+S_{BC}-S_{ABC}-S_A-S_B-S_C
\end{aligned} \tag{5.20}$$

where in the last line, we used three party subadditivity ($S_{ABC} \leq S_A + S_B + S_C$). We recognize the last expression above as $I(A:B:C)$, thus completing the proof.

Upper bounding holographic cyclic information

Similarly, the following upper bound on C_k can also be derived:

$$\sum_{i=1}^{n} E_W^G(A_i, A_{i+i}, \ldots, A_{i+k} : A_{i+k}, A_{i+k+1}, \ldots A_{i+n-1}) \geq C_k(A_1, \ldots, A_n), \tag{5.21}$$

where, as before, the indices are to be interpreted mod n, and $n = 2k+1$.

To prove this, we use Eq. (5.14) to get:

$$\begin{aligned}
&\sum_{i=1}^{n} E_W^G(A_i, \ldots, A_{i+k} : A_{i+k}, \ldots A_{i+n-1})\\
&\geq \sum_{\text{cyc}} S(A_1 \cdots A_{k+1}) - \frac{n}{2}S(A_1A_2\ldots A_n) - \frac{1}{2}\sum_{j=1}^{n} S(A_j)\\
&\geq \sum_{\text{cyc}} S(A_1 \cdots A_{k+1}) - S(A_1A_2\ldots A_n) - \frac{n-2}{2}S(A_1A_2\ldots A_n) - \frac{1}{2}\sum_{j=1}^{n} S(A_j)\\
&\geq C_k(A_1\ldots A_n),
\end{aligned} \tag{5.22}$$

where we have used subadditivity and that

$$2\sum_{\text{cyclic}} S(A_1 \ldots A_k) \geq (n-2)S(A_1 \ldots A_n) + \sum_j S(A_j), \tag{5.23}$$

which follows from repeated application of SSA, as we now show. First, pairwise application of SSA to terms of the form $S(A_i, \ldots, A_k)$ and $S(A_k, \ldots, A_{2k-1})$ on the left-hand side gives:

$$2\sum_{\text{cyclic}} S(A_1 \ldots A_k) \geq \sum_j S(A_j) + \sum_{\text{cyc}} S(A_1 \ldots A_{2k-1}) \tag{5.24}$$

Now, let F be a purification of $A_1 \dots A_n$, so that we have $\sum_{\text{cyc}} S(A_1 \dots A_{2k-1}) = \sum_i S(A_i A_{i+1} F)$. Applying SSA now to $S(A_{2i}A_{2i+1}F)$ for $i = 1, \dots, k$, and to $S(A_{2i-1}A_{2i}F)$ for $i = 1, \dots k$ we get

$$\begin{aligned}\sum_i S(A_i A_{i+1} F) &\geq (n-1)S(F) + S(A_1 A_n F) + S(A_1 \dots A_{n-1} F) + S(A_2 \dots A_n F) \\ &\geq (n-2)S(F) = (n-2)S(A_1 \dots A_n). \quad (5.25)\end{aligned}$$

Finally, pairwise application of SSA to $S(A_i \dots A_{i+k+1})$ and $S(A_{i+k+1} \dots A_{i+n-1})$ for $i = 1$ to k gives

$$\sum_{\text{cyc}} S(A_1 \dots A_{k+1}) \geq kS(A_1 \dots A_n) + \sum_{i=1}^{k} S_i + S(A_{k+1}A_{k+2} \dots A_n) \geq (k+1)S(A_1 \dots A_n). \tag{5.26}$$

Combining Eqs. (5.24), (5.25), and (5.26) yields Eq. (5.23).

Cyclic E_W Inequalities

Here we use as a starting point the cyclic entropy inequalities, Eq. (5.8), to derive cyclic E_W inequalities. Interestingly, as we show in section 5.4, the inequalities we arrive are not obviously violated for generic quantum states when E_W is replaced by E_p.

We first rewrite the cyclic entropy cone inequalities, Eq. (5.8), in terms of only mutual information as

$$\sum_{i=2}^{n} I(A_1 A_2 \dots A_{i-1} : A_i) \geq \sum_{\text{cyclic}} I(A_1 : A_2 \dots A_{1+k}). \tag{5.27}$$

Then, by the upper bound in Eq. (5.3), the left-hand side of the inequality above can be upper bounded by a combination of E_W's, which gives

$$\sum_{i=2}^{n} E_W(A_1 A_2 \dots A_{i-1} : A_i) \geq \frac{1}{2} \sum_{\text{cyclic}} I(A_1 : A_2 \dots A_{1+k}). \tag{5.28}$$

5.4 Bounding entanglement entropy with E_p

Generalized E_p

Just as we did for E_W, we will similarly need to generalize E_p. The **generalized entanglement of purification** E_p^G is defined as[4]

$$E_p^G(A:B) = \min_{A'B'C^{(A)}} S((A \backslash B)A'C^{(A)}), \tag{5.29}$$

[4] A different generalization was proposed in [99].

where as before we require that $AA'BB'$ is pure, and we now also require that $C^{(A)} \subset C \equiv A \cap B$. For convenience, we also define $C^{(B)} = C \backslash C^{(A)}$ (see Fig. 5.3). Note that the minimization could also have been done over $S((B\backslash A)B'C^{(B)})$ and also that if $A \cap B = \emptyset$ then $E_p^G(A:B) = E_p(A:B)$. Moreover, even when A, B and C have a geometrical interpretation (as is the case when they have a holographic bulk dual), there is no requirement that the split of C into $C^{(A)}$ and $C^{(B)}$ be geometric.

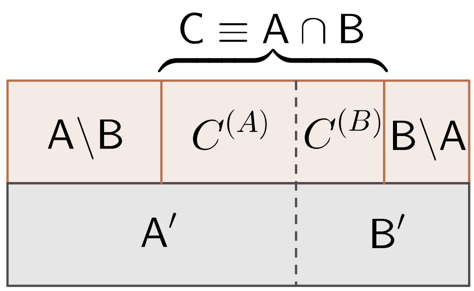


Figure 5.3: As depicted, for the optimal choices of A', B' and $C^{(A)}$, we have $E_p^G(A:B) = S((A\backslash B)A'C^{(A)})$.

E_p^G obeys known E_p inequalities

We now show that E_p^G also obeys the known inequalities for E_p.

The upper bound in Eq. (5.3) follows from

$$\begin{aligned} E_p^G(A:B) &\leq E_p(A:B\backslash A) \leq \min(S(A), S(B\backslash A)) \text{ and} \\ E_p^G(A:B) &\leq E_p(A\backslash B:B) \leq \min(S(A\backslash B), S(B)), \end{aligned} \tag{5.30}$$

where the first inequality in each line follows from the fact the minimization procedure defining $E_p^G(A:B)$ is less constrained than the one defining $E_p(A:B\backslash A)$ or $E_p(A\backslash B:B)$[5], and the second inequality in each line follows from Eq. (5.3). Together these imply

$$E_p^G(A:B) \leq \min(S(A), S(B)). \tag{5.31}$$

The lower bound in Eq. (5.3) follows from

$$\begin{aligned} E_p^G(A:B) &\geq E_p(A\backslash B:B\backslash A) \\ &\geq \frac{1}{2}\left(S(A\backslash B) + S(B\backslash A) - S(A\backslash B \cup B\backslash A)\right) = I^G(A:B), \end{aligned} \tag{5.32}$$

[5]This is because for $E_p^G(A:B)$ we are free to choose $C^{(A)}$, while in $E_p(A:B\backslash A)$ we have that $C^{(A)} = \emptyset$, and in $E_p(A\backslash B:B)$ we have $C^{(A)} = C = A \cap B$.

where we have used the fact that the minimization procedure for $E_p^G(A:B)$ is more constrained that the one for $E_p(A\backslash B : B\backslash A)$[6] and Eq. (5.3).

For any $C \cap A = \emptyset$, it is easy to see that

$$E_p^G(A:BC) \geq E_p(A:B), \text{ and} \tag{5.33}$$

$$E_p^G(A:BC) \geq E_P(A-B,(B-A)C) \geq \frac{1}{2}\left(I^G(A:B) + I(A\backslash B : C)\right) \tag{5.34}$$

since adding C further constrains the optimization.

Upper bounding conditional mutual information

We now prove the following upper on conditional mutual information:

$$I(A:B|C) \leq 2E_p^G(AC:BC). \tag{5.35}$$

Note that, similarly to the E_W^G bound, in the case where $C=\emptyset$, this reduces to $I(A:B) \leq 2E_p(A:B)$.

Assume $A \cap B = \emptyset$, and let A', B', and $C^{(A)} \subset C$ define an optimal purification of $(AC:BC)$ according to Eq. (5.29). Then, we can get the desired upper bound by repeated application of strong subadditivity:

$$\begin{aligned}
2E_p^G(AC:BC) + S(ABC) + S(C) &= S(AA'C^{(A)}) + S(BB'C^{(B)}) + S(ABC) + S(C) \\
&\geq S(AA'BC) + S(AC^{(A)}) + S(BB'C^{(B)}) + S(C^{(A)}C^{(B)}) \\
&\geq S(AA'BC) + S(BB'C^{(B)}) + S(C^{(A)}) + S(AC) \\
&\geq S(BC^{(B)}) + S(AC) + S(C^{(A)}) \geq S(BC) + S(AC).
\end{aligned} \tag{5.36}$$

Upper bounding tripartite information and cyclic information

It is worth noting that the proofs in the previous section of upper bounds for holographic tripartite information, Eq. (5.19), and holographic cyclic information, Eq. (5.21), depended only on Eq. (5.14). Since the analogous statement obtained by replacing E_W^G by E_p^G, i.e., Eq. (5.35), also holds, the E_p^G versions of these upper bounds are also true.

Lower bound on tripartite information

One can also extract a quantum lower bound for the tripartite information:

$$I(A:B:C) \geq -2E_p(A:BC) - 2E_p(B:C). \tag{5.37}$$

[6]This follows from the fact that we can always take $C^{(A)}$ to be part of A' and $C^{(B)}$ part of B'.

This inequality is obviated in the holographic context by positivity of holographic tripartite information. For a general quantum state, however, it is nontrivial. To prove this inequality, we add three instances of positivity of conditional mutual information to find:

$$\begin{aligned} &I(A:B|C)+I(A:C|B)+I(B:C|A) \\ &= 2S(AB)+2S(BC)+2S(AC)-S(A)-S(B)-S(C)-3S(ABC) \geq 0. \end{aligned} \tag{5.38}$$

We can add to this inequality the inequality

$$S(ABC)-S(A)-S(B)-S(C) \geq -2E_p(A:BC)-2E_p(B:C), \tag{5.39}$$

which follows from two applications of Eq. (5.3). The sum of Eqs. (5.38) and (5.39) proves the lower bound in Eq. (5.37).

Cyclic E_p inequalities

If the $E_W = E_p$ conjecture is correct, then for holographic states, it follows from Eq. (5.28) that

$$\sum_{i=2}^{n} E_p(A_1A_2\ldots A_{i-1}:A_i) \geq \frac{1}{2}\sum_{\text{cyclic}} I(A_1:A_2\ldots A_{1+k}) \tag{5.40}$$

However, it is interesting to note that in deriving this, we have combined several inequalities, thereby weakening them. For instance, the GHZ state defined by $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n}+|1\rangle^{\otimes n}\right)$ is not holographic for $n \geq 4$ and violates instances of Eq. (5.8), but still satisfies Eq. (5.28). This can be seen from the fact that for any A and B disjoint proper subsystems of GHZ, we have[9]:

$$E_p(A:B) = S(A) = S(B), \text{ and } I(A:B) = S(A) = S(B). \tag{5.41}$$

Thus all the terms in Eq (5.27) and Eq. (5.40) are the same, and we can see the former is violated, while the latter satisfied.

Thus, it is plausible that the inequalities in Eq. (5.40) hold for general quantum states. Because random states are known to obey the holographic inequalities [86], it is also clear that those states would also obey Eq. (5.40).

We now present evidence that these inequalities are also obeyed by W states, which are also known not to be holographic and are defined by

$$|W\rangle = \frac{1}{\sqrt{n}}\left(|100\ldots0\rangle+|010\ldots0\rangle+\cdots|00\ldots01\rangle\right). \tag{5.42}$$

For any qubit system invariant under the permutation of qubits, we have

$$\sum_{\text{cyclic}} I(A_1:A_2\ldots A_{1+k}) = n(S(\rho_1)+S(\rho_k)-S(\rho_{k+1})), \tag{5.43}$$

where ρ_i is the reduced density matrix for the i-qubit subsystem. Moreover, by using Eq. (5.5) and permutation symmetry, we can lower bound the left-hand side of Eq. (5.40) as follows:

$$\sum_{i=2}^{n} E_p(A_1A_2\dots A_{i-1}:A_i) \geq \frac{1}{2}\left(6kS(\rho_1) - 2kS(\rho_2) - S(\rho_{2k})\right) \tag{5.44}$$

.

Thus, Eq. (5.40) is implied by

$$D \equiv (4k-1)S(\rho_1) - 2k(S(\rho_2)) - S(\rho_{2k}) - (2k+1)S(\rho_k)(2k+1) + S(\rho_{k+1}) \geq 0. \tag{5.45}$$

Let W_n be the density matrix for the n qubit W state, and let $W_{n,k}$ be its reduced density matrix to a k qubit subsystem. We can now write these in component form as

$$W_n = \frac{1}{n}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.46}$$

where row i column j contains a 1 if and only if $i-1$ and $j-1$ are powers of 2, and

$$W_{n,k} = \frac{1}{n}\begin{pmatrix} n-k & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.47}$$

where, apart from the first entry, the same pattern is followed. This allows us to evaluate the left-hand side of Eq. (5.45) and verify its positivity (for numerically tractible n and k). Moreover, the best fit we found for these curves indicate that this is satisfied for any value of k and n (See Figure 5.4).

5.5 Future direction: new dictionary entries

In this section, we use intuition from bit threads [60] and from the fact that for states with a holographic dual, $E_p = E_p^{\infty} = E_{LOq}$ [96] to strengthen the $E_W = E_p$ conjecture. For a

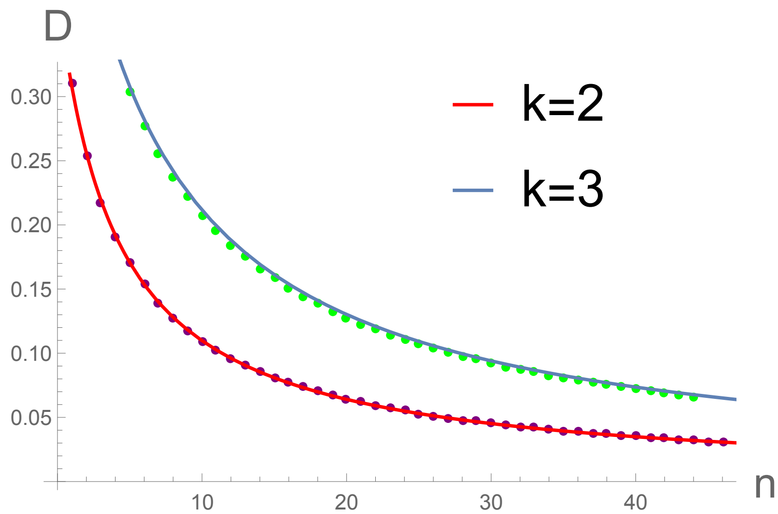


Figure 5.4: Displaying the left-hand side of Eq. (5.45) for $k=2$ and $k=3$ for W_n as a function of n, as well as the best fit curves of the form $D=\frac{B}{n}$. For $k=2$, we found $B\approx 1.537$, and for $k=3$, we found $B\approx 3.385$.

definition of these quantities see [97]. For our purposes, we will mainly use the fact that $E_{LOq}(\rho_{AB})$ is roughly equal to the number of EPR pairs needed to get to ρ_{AB} by means of only local operations.

Let A and B be boundary regions, and let the state on it be given by the density matrix ρ_{AB}. Consider the maximum flow Φ from A and into B [7]. We interpret these as bit threads connecting EPR pairs living on the boundary, and we assume the global boundary state to be pure [8]. We can now state our conjecture:

Conjecture Any (non-minimal) surface Γ that partitions r_{AB} into a region homologous to A and a region homologous to B is dual to a (suboptimal) purification $A'B'$ such that qubits on the boundary are in A' if they are connected to A without crossing Γ or are connected to B while crossing Γ. Likewise, boundary qubits are in B' if they are connected to B without crossing Γ, or connected to A while crossing Γ.

The conjecture gives EPR pairs from which the original ρ_{AB} can be reached by local operations on A and on B. Note that the purifying system of AB dual to Γ would be $A'B'$ and $S(AA')=\Phi(\Gamma)$, the flow Φ across Γ. It is clear then that $S(AA')$ is minimized when Γ is the entanglement wedge cross-section.

Combining our conjecture with max-flow min-cut implies the $E_W=E_p$ conjecture[9], as

[7]We can do this by simultaneously maximizing the flow out of both A and B^c, which contains A, something permitted by the nesting property of bit threads.

[8]This can always be achieved via multiboundary wormhole completion.

[9]In turn, the $E_W=E_p$ conjecture implies Ryu-Takayanagi as a special case.

the area of the minimal cross-section of the entanglement wedge is given by the number of bit threads crossing it in this construction, due to its nature as a bottleneck for the flow from A to B. Thus, it is clear that for any cut Γ there exist an A' and B' such that $\Phi(\Gamma) = S(AA')$.

Still, the conjecture places nontrivial constraints on the dimensionality of the purifying system. This is because the sum of the number of bit threads emerging from the A system when maximizing the flow through A and those emerging from the B system when maximizing the flow through B upper bounds the log of the dimensionality of the $A'B'$ system. Thus, we get

$$\log d_{A'B'} \leq S(A) + S(B). \tag{5.48}$$

Note that this is much tighter than the upper bound given in [97]; this is not all that surprising, however, given that holographic states have much less entanglement than the generic quantum states considered in [97]. Moreover the overall holographic state is pure, and thus one does not get confounding entanglement of purification from considering classical mixtures.

It would be interesting to study the plausibility of this conjecture in toy models of holography, in particular suitable generalizations of the qutrit code [6], perfect tensors [80], or the random tensor model [57]. In these models one would be able to falsify our conjecture by finding a system AB for which there is no optimal purification $A'B'$ with dimension $d_{A'B"}$ satisfying Eq.(5.48). In these simpler models, it may also be possible to explicitly reconstruct A' and B' from given known A and B, say by explicitly searching for the unitary that would extract either the A' or B' systems tensored with unentangled ancilla qubits from the systems in which they are conjectured to be contained; because the search space is much smaller in finite dimensional systems, this search is in principle feasible here.

Other interesting directions of future research include to either prove as disprove Eq. (5.40) as an inequality valid for all quantum systems, and to extend the results of the present paper to the fully covariant case.

5.6 Conclusion

In this paper, we have considered upper and lower bounds for several information theoretic quantities, including bounds on the conditional mutual information, tripartite information, and cyclic information. Despite being motivated by holography, we have shown these to hold for all quantum states. We have also found a new family of holographic inequalities for E_W, and provided evidence that the corresponding inequality for E_p may be true for all quantum states. These results are summarized in Table 5.1.

Finally, we conjectured a potential extension of the $E_W = E_p$ conjecture of [96, 76], which asserts that all cuts of the entanglement wedge are dual to purifications. If true, it may be possible to write such a map explicitly, which could lead to an efficient way of computing entanglement of purification.

	Lower Bound	Upper Bound
Mutual Information I(A:B)	$\mathbf{0}$	$\mathbf{2E_p(A:B)}$
Conditional Mutual Information I(A:B—C)	$\mathbf{0}$	$\boldsymbol{2E_p^G(AC:BC)}$
Tripartite Information I(A:B:C)	0, and $\boldsymbol{-2E_p(AB:C)-2E_p(B:C)}$	$\boldsymbol{E_p^G(AC:BC)+E_p^G(AB:BC)}$ $\boldsymbol{+E_p^G(AB:AC)}$
Cyclic Information $C_k(A_1,\ldots,A_{2k+1})$	0	$\boldsymbol{\sum_{i=1}^n E_p^G(A_i,\ldots,A_{i+k}:A_{i+k},\ldots A_{i+n-1})}$

Table 5.1: The main results are listed in the table. In $\boxed{\text{standard black font}}$ are the results that only hold for holographic states. All others hold for general quantum states. Results that, to the best of the authors knowledge, are new are in ***red bold italics text*** (the upper bound on conditional mutual information was proved for E_W in the language of bit threads in [60]). In **blue bold text** are the inequalities that were already known to hold for all quantum states. In addition to these results, we have also shown the holographic inequality in Eq. (5.28), which does not fit neatly into this table. The general version of this inequality, pending the $E_W = E_p$ conjecture, is given by Eq. (5.40). We have shown this to hold for several non-holographic states, but its general validity is still an open question.

Chapter 6

Conditional and Multipartite Entanglements of Purification and Holography

6.1 Introduction

There has been much progress made in recent years at the intersection of quantum information and quantum gravity. One particular area of impact is the study of entanglement entropy in the context of holography pioneered by [89]. In this context, it was discovered that entanglement entropies for states in holographic theories dual to a classical bulk geometry obeyed additional inequalities beyond those obeyed by all quantum states [56, 15] and that such states are a small fraction of the set of all quantum states in the entropy space measure [10]. These inequalities, though not true for all quantum systems, therefore serve as a useful discriminator for which quantum states, even in theories known to possess a holographic duality, can be dual to (semi)-classical spacetimes.

It is therefore a natural question to ask whether other entanglement measures are also dual to objects in holography. Recently, it has been conjectured by [96, 76] that the entanglement of purification (E_p) [97] is dual to an object called the entanglement wedge cross-section (E_W). This conjecture ($E_p = E_W$) powerfully suggests that the holographic state is an optimal purification[1] of the density matrix of any geometric subregion of the boundary theory. In further work [12], it was shown that there exists a conditional generalization of the entanglement of purification (with a corresponding holographically dual object) that passes the same consistency checks as the $E_p = E_W$ conjecture. This conditional generalization in the holographic context suggests an interpretation of the portion of the entanglement wedge of a region ABC excluding the entanglement wedge of a subregion C as being related to a constrained purification of the density matrix ρ_{AB} given that the purification must include

[1]Here we mean optimal in the sense of satisfying the minimization constraint that defines the entanglement of purification.

C. Moreover, this conditional entanglement of purification can be shown to nontrivially upper bound the conditional mutual information in any quantum state.

In a similar spirit, one can ask if there exists a simple generalization of the entanglement of purification that would upper bound other multipartite entanglement combinations such as the tripartite information, shown to be positive in [56] [2]. In this work, we will show that the answer to this question is yes. In fact, we find generalizations of the entanglement of purification that upper bound both the tripartite information and the cyclic combinations shown to be positive holographically in [15]. Indeed, we prove that these upper bounds hold in any quantum system, regardless of the existence of a holographic dual.

After the first version of this paper appeared, a definition of multipartite entanglement of purification differing from ours only by a factor was proposed in [96], where a conjectured holographic dual which differs from our multipartite generalization of E_W was also proposed.

The organization of this paper is as follows. In section 2, we briefly review the definitions and properties of the standard entanglement of purification and entanglement wedge cross-section. In section 3, we review the definition of the conditional entanglement of purification, its conjectured holographic dual, and demonstrate a few new properties of both. In section 4, we define the multipartite entanglement of purification and multipartite entanglement wedge cross-section, and prove they they share several properties but, nonetheless, are not holographic duals. Finally, we conclude with some discussion in section 5.

6.2 Preliminary Definitions

Consider a bipartite quantum system $AB = A \cup B$, where A and B are taken to be disjoint. In fact, unless of otherwise stated, any two regions will be taken to be disjoint throughout the paper. The entanglement of purification $E_p(A : B)$ is defined by

$$E_p(A : B) = \min\{S(AA'); \rho_{AA'BB'} \text{ is pure}\} \tag{6.1}$$

where S is the Von Neumann entropy. E_p is known to satisfy the following inequalities [97, 9]:

$$\min(S_A, S_B) \geq E_p(A : B) \geq \frac{1}{2} I(A : B) \tag{6.2}$$

$$E_p(A : BC) \geq E_p(A : B) \tag{6.3}$$

$$E_p(AB : C) \geq \frac{1}{2}\left(I(A : C) + I(B : C)\right), \tag{6.4}$$

where $I(A : B) \equiv S(A) + S(B) - S(AB)$ is the mutual information between A and B.

The $E_p = E_W$ conjecture was motivated by the proofs in [96] that the above inequalities are also all satisfied by a holographic object, the entanglement wedge cross-section, E_W, defined by:

[2] In [12] this was done using combinations of conditional entanglements of purifications, but that bounding was not tight.

$$E_W(A:B) = \min\{\text{Area}(\Gamma); \Gamma \subset r_{AB} \text{ splits } r_{AB} \text{ into two regions homologous respectively to } A \text{ and } B\}, \tag{6.5}$$

where r_{AB} is the restriction of the entanglement wedge[60] of AB to some time-symmetric slice. This restriction will be left implicit in what follows. Also implicit in this definition, and any holographic statement, is the existence of a (semi)-classical bulk geometry. The concepts of E_p and E_W are illustrated in Figure (6.1).

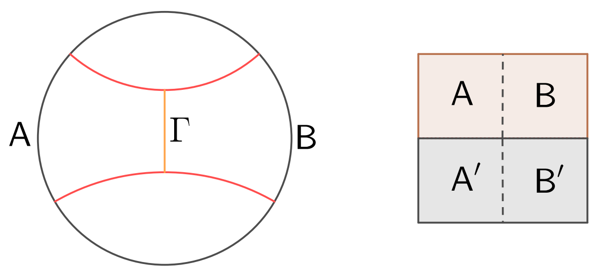


Figure 6.1: To the left, $A'B'$ purifies AB. For a choice of A' and B' over all such purifying systems that minimizes the entanglement across the dashed partition we have $E_p(A:B) = S(AA')$. To the right, Γ is the minimal surface the separates the entanglement wedge cross-section of AB into a region homologous to A and a region homologous to B. Its area is $E_W[A:B]$. [Figure adapted with permission from figure in [12]].

6.3 The Conditional Entanglement of Purification

In this section, we interpret quantities previously defined in [12] as conditional entanglement of purification[3] and conditional entanglement wedge cross-section. We also derive some new properties.

When conditioned on subsystem C, we get the **conditional entanglement of purification** defined by

$$E_p(A_1 : A_2|C) = \min_{A_1'A_2'C^{(1)}} \left\{S(A_1A_1'C^{(1)}), \text{s.t. } \rho_{A_1A_2A_1'A_2'C} \text{ is pure and } C^{(1)} \subset C\right\}, \tag{6.6}$$

[3] In [12], what we now write as $E_p(A:B|C)$ was denoted $E_p(AC:BC)$, and likewise for E_W. This new notation is to standardize with the new conditional interpretation. While this article was in preparation, reference [44] appeared using a notation suggestive of the conditional interpretation.

and the **conditional entanglement wedge cross-section** by

$$E_W(A_1 : A_2|C) = \min_{\Gamma \in r(A_1 A_2 C) \backslash r(C)} \{\text{Area}(\Gamma), \text{s.t. } \Gamma \text{ splits } r(A_1 A_2 C) \backslash r(C) \text{ accordingly}\} . \tag{6.7}$$

In the spirit of the new conditional interpretation, one can also prove conditional analogs of (Eqs. (6.2)–(6.4)); the first two of these were proven in [12], but the third was missed there due to [12] not referencing the conditional interpretation. Nevertheless, it is straightforward to show using the techniques of [12] that its conditional generalization holds, i.e.

$$E(A : BC|D) \geq \frac{1}{2} I(A : B|D) + \frac{1}{2} I(A : C|D), \tag{6.8}$$

where E here can stand for either E_p orE_W.

Furthermore, the following inequality may be dubbed the super-Bayesian property[4]:

$$E_W(A_1 B_1 : A_2 B_2|C) \geq E_W(A_1 : A_2|BC) + E_W(B_1 : B_2|C), \tag{6.9}$$

where $B = B_1 \cup B_2$. The name is due to the resemblance with the Bayesian property of probabilities:

$$\ln p(AB|C) = \ln p(A|BC) + \ln p(B|C). \tag{6.10}$$

It is easy to see that E_W is super-Bayesian, i.e., it satisfies Eq. (6.9). This follows from the fact that the minimal surface that splits $A_1 B_1$ from $A_2 B_2$ in $r(ABC) \backslash r(C)$ can be broken into a piece that splits A_1 from A_2 in $r(A) \backslash r(BC)$ and one that splits B_1 from B_2 in $r(BC) \backslash r(C)$, and that a constrained optimizations is at most as optimal as a strictly less constrained optimization. See figure 6.2.

On the other hand, E_p is not super-Bayesian for arbitrary quantum states. For instance, one can construct a counterexample by choosing the regions to be subsystems of of GHZ states [53]. If the $E_p = E_W$ conjecture is correct, however, E_p must be super-Bayesian for any holographic state dual to a classical bulk geometry. This therefore allows for the super-Bayesian property to be used in an analogous way to the holographic entanglement entropy inequalities, i.e., as a discriminator of which states can be dual to (semi)-classical bulk geometries.

6.4 The Multipartite Entanglement of Purification

In this section, we define the multipartite entanglement of purification $E_p(A_1 : A_2 : \cdots : A_n)$ and the multipartite entanglement wedge cross-section $E_W(A_1 : A_2 : \cdots : A_n)$. Both of these quantities reduce to the bipartite objects when $n = 2$, and obey inequalities which reduce to some of those that motivated the formulation of the $E_p = E_W$ conjecture (Eqs. (6.2)–(6.4)). This initially led the authors of this paper to conjecture that they were holographic duals. However, in section 6.4 we show this not to be the case.

[4] Note this is also a generalization of the "strong super-additivity" inequality from [96].

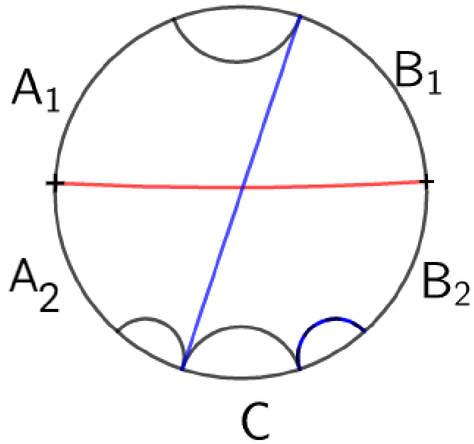


Figure 6.2: The red line is the minimal surface splitting A_1B_1 from A_2B_2 in $r(ABC)\backslash r(C)$, which is the region bounded by black lines. The RT surface of BC is displayed in blue. It clearly splits the red line into two, one that splits (not necessarily minimally) A_1 from A_2 in $r(ABC)\backslash r(BC)$ and one that splits (not necessarily minimally) B_1 from B_2 in $r(BC)\backslash r(C)$.

The **multipartite entanglement of purification** is defined by

$$E_p(A_1 : A_2 : \cdots : A_n) = \min_{A'} \left\{ \frac{1}{n} \sum_{i=1}^{n} S(A_i A_i'), \text{such that } \rho_{AA'} \text{ is pure} \right\}, \tag{6.11}$$

where $A = \cup_i A_i$ and $A' = \cup_i A_i'$. (See Fig. 6.3 for an example). It is clear that when $n = 2$, we get back the usual $E_p(A_1 : A_2)$.

Likewise, we define the **multipartite entanglement wedge cross-section** $E_W(A_1 : A_2 : \cdots : A_n)$ as

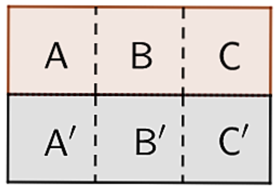
$$E_W(A_1 : A_2 : \cdots : A_n) = \min_{\Gamma \in r(A)} \left\{ \frac{2}{n} \text{Area}(\Gamma), \text{s.t. } \Gamma \text{ splits } r(A) \text{ into } n \text{ regions homologous to each } A_i \right\}. \tag{6.12}$$

See Fig. 6.3 for an example with three regions.

We now study whether the inequalities (6.2)–(6.4) obeyed by E_p and E_W in the bipartite ($n = 2$) case can be extended to the general multipartite case. Equations (6.2) and (6.3) can be generalized in the multipartite case to

$$\frac{2}{n} \left(\sum_{i=1}^{n} S(A_i) - \max_i S(A_i) \right) \geq E(A_1 : \cdots : A_n) \geq \frac{1}{n} C_k(A_1 : \cdots : A_n), \text{ and} \tag{6.13}$$

$$E(A_1 : A_2 : \cdots : A_i B : A_{i+1} : \cdots : A_n) \geq E(A_1 : A_2 : \cdots : A_i : A_{i+1} : \cdots : A_n), \tag{6.14}$$



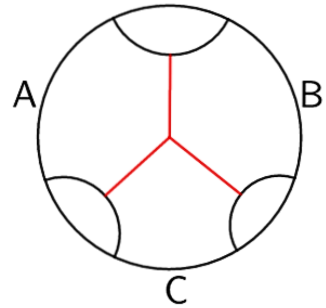


Figure 6.3: On the left, $A'B'C'$ purifies ABC while minimizing $\frac{1}{3}\left(S(AA') + S(BB') + S(CC')\right)$. This minimal value is $E_p(A : B : C)$. On the right, the red surface is the minimal surface separating $r(ABC)$ into three regions, one homologous to A, one to B and one to C. Its area is $\frac{3}{2}$ of $E_W(A : B : C)$.

where $n = 2k + 1$ and C_k is the cyclic information [12, 15]defined as

$$C_k(A_1 : \dots : A_n) \equiv \sum_{i=1}^{n} \left(S(A_i A_{i+1} \dots A_{i+k}) - S(A_i)\right) - S(A_1 \dots A_n), \tag{6.15}$$

where indices are interpreted mod n.

We will show these to hold for both $E = E_p$ and $E = E_W$. When $k = 1$, the cyclic information reduces to the tripartite information, I_3, and equation (6.13) provides a novel way of upper bounding it. One might also try to generalize Eq.(6.4) to the multipartite case as follows:

$$E(A_1 : \dots : A_i B : \dots : A_n) \geq \frac{1}{n}\left(C_k(A_1 : \dots : A_i : \dots : A_n) + C_k(A_1 : \dots B : \dots : A_n)\right). \tag{6.16}$$

However, as of now, we have not been able to prove Eq. (6.16) for arbitrary n. We will, nonetheless, prove it for $n = 3$, which gives us another inequality involving tripartite information:

$$E(A : BC : D) \geq \frac{1}{3}\left(I_3(A : B : D) + I_3(A : C : D)\right). \tag{6.17}$$

Proof of inequalities for multipartite E_p

To show the upper bound in Eq. (6.13), consider without loss of generality that $S(A_n) = \max_i S(A_i)$, and let $A' = A'_n$, i.e., consider a purification with $A_i = \emptyset$ for $i \leq n-1$. Then, we get

$$\begin{aligned} E_p(A_1 : \cdots : A_n) \leq \frac{1}{n}\left(\sum_{i=1}^{n-1} S(A_i) + S(A_n A_n')\right) &= \frac{1}{n}\left(\sum_{i=1}^{n-1} S(A_i) + S(A_1 A_2 \dots A_{n-1})\right) \\ &\leq \frac{2}{n}\sum_{i=1}^{n-1} S(A_i), \end{aligned} \tag{6.18}$$

where we have used that AA' is pure, and subadditivity. To show the lower bound in Eq. (6.13), consider an optimal purification $A' = \cup_i A_i'$. Then,

$$\begin{aligned} & nE_p(A_1 : \cdots : A_n) + \sum_i S(A_i \dots A_{i+k}) + S(A_1 \dots A_n) \\ = & \sum_{i=1}^{n} S(A_i A_i') + \sum_i S(A_i \dots A_{i+k-1}) + S(A_1 \dots A_n) \\ = & \sum_{i=1}^{n} [S(A_i A_i') + S(A_{i+1} \dots A_{i+k})] + S(A_1 \dots A_n) \\ \geq & \sum_{i=1}^{n} S(A_i' A_i A_{i+1} \dots A_{i+k}) + S(A_1 \dots A_n) \\ \geq & \sum_{i=2}^{n} S(A_i' A_i A_{i+1} \dots A_{i+k}) + S(A_1 A_2 \dots A_N A_1') + S(A_1 \dots A_{1+k}) \\ \geq & \sum_{i=1}^{n} S(A_i A_{i+1} \dots A_{i+k}) + S(AA') = \sum_{i=1}^{n} S(A_i A_{i+1} \dots A_{i+k}) \end{aligned} \tag{6.19}$$

where indices are mod n, and we have used strong subadditivity and the fact that AA' is pure. Rearranging the terms gives the lower bound for E in Eq. (6.13).

Monotonicity, Eq. (6.14), follows from the fact that the quantity on the right-hand side of the inequality is defined as the solution of a less constrained optimization problem. Since B can be considered as part of A_k', any purification of AB is also a purification of A. Interestingly, when $k = 1$, this gives a novel bound on the tripartite information $I_3 = C_1$.

Let's now show that tripartite E_p satisfies Eq. (6.17). Let

$$E(A : BC : D) = \frac{1}{3}\left(S(AA') + S((BC)(BC)') + S(DD')\right)$$

for some purification. Then,

$$\begin{aligned}
& 3E(A:BC:D)+2S(A)+S(B)+S(C)+2S(D)+S(ABD)+S(ACD) \\
= \;& S(AA')+S((BC)(BC)')+S(DD')+2S(A)+S(B)+S(C)+2S(D)+S(ABD)+S(ACD) \\
\geq \;& S(AA')+S((BC)(BC)')+S(DD')+S(AC)+S(AD)+S(BD)+S(ABD)+S(ACD) \\
\geq \;& S(AA')+S((BC)(BC)')+S(AB)+S(D')+S(AC)+S(AD)+S(BD)+S(ACD) \\
\geq \;& S(A')+S(D')+S((BC)(BC)')+S(AB)+S(AC)+S(AD)+S(BD)+S(CD) \\
\geq \;& S(A'D')+S(ADA'D')+S(AB)+S(AC)+S(AD)+S(BD)+S(CD) \\
\geq \;& S(AD)+S(CD)+S(AB)+S(AD)+S(BD)+S(AC),
\end{aligned} \tag{6.20}$$

where we have used subadditivity, weak monotonicity, and that $\rho_{ABCDA'(BC)'D'}$ is pure. A rearrangement of the above inequality gives (6.17).

Proof of inequalities for multipartite E_W

The upper bound in Eq. (6.13) can be established by noticing that the union of $(n-1)$ Ryu-Takayanagi (RT) surfaces splits $r(A)$ into the desired n regions. Picking the $(n-1)$ RT surfaces with the smallest areas gives us this bound. For a pictorial proof of the lower bound, see Figure 6.4.

Equation (6.14) holds because, by entanglement wedge nesting [102, 5], we have that $r(AB) \supset r(A)$, and thus if a surface Γ splits $r(AB)$ into n regions homologous to each of $A_1, A_2, \ldots, A_iB, \ldots, A_n$, then $\Gamma \cap r(A)$, which can have no greater area than Γ, will split $r(A)$ into n regions homologous to each of $A_1, A_2, \ldots, A_i, \ldots, A_n$.

Even though we do not know of a proof of the more general Eq. (6.16), we can prove Eq. (6.17) holographically by following line by line Eq. (6.20). This is not particularly illuminating, but it can be made rigorous using inclusion-exclusion techniques as in [56].

Since multipartite E_p, as defined by Eq. (6.11), and multipartite E_W, as defined by Eq. (6.12), obey the same set of inequalities, one may be led to believe these to be duals. Indeed, in a previous version of this paper, the authors conjectured this to be the case. However, a holographic tripartite pure state is a counterexample: if ρ_{ABC} is pure, it follows that $E_p(A:B:C) = \frac{1}{3}(S(A)+S(B)+S(C))$ [96], while the corresponding E_W is generically larger than that (See Fig. 6.5). The geometric object conjectured in [96] as dual to the multipartite entanglement of purification evades this counterexample, and it remains to see if it will endure future tests.

Given that our multipartite generalization of the entanglement wedge is not dual to our multipartite generalization of the entanglement of purification, a natural question arise as to what it is dual to. Multipartite E_W, as defined by Eq.(6.12), is a geometrical object obeying nontrivial inequalities and with the property of being able to probe features of the entanglement wedge that lie outside of the causal wedges. As such, we believe that it would be an interesting future work to determine what information theoretic quantity they are dual to.

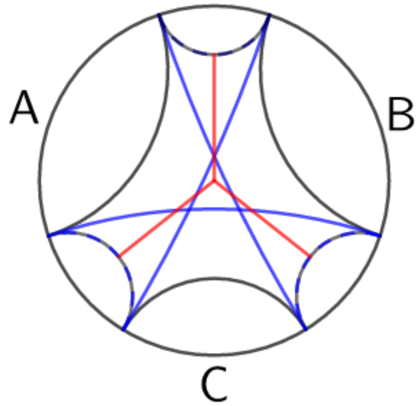


Figure 6.4: Visual proof of the lower bound in Eq. (6.13) for E_W shown for $k = 1$. The proof generalizes straightforwardly to arbitrary k. Rearranging Eq. (6.13) so that terms on both sides of the inequality are all positive, we get $3E_W(A : B : C) + S(A) + S(B) + S(C) + S(ABC) \geq S(AB) + S(AC) + S(BC)$. The black and the red lines correspond to the greater than (or equal) side of the inequality, with the red lines corresponding to the $3E_W$ term and being doubled (see definition of E_W in Eq. (6.12)). The blue lines correspond to the lesser than (or equal) side of the inequality. The dashed black-and-blue lines appear on both sides. By subadditivity, $S(A) + S(B) + S(C) \leq S(ABC)$, allowing us to replace the red lines with the dashed black-and-blue lines. Using these and each red segment twice, one can subtend each blue arc sub-optimally.

6.5 Discussion

The multipartite and conditional entanglements of purification provide nontrivial upper bounds to known information theoretic quantities. In particular, in the context of the tripartite and cyclic informations, they give them new, Holevo-like [63], interpretations as the optimal multipartite entanglement of purification of some density matrix in any quantum system [5]. Moreover, the fact that the conditional $E_p = E_W$ conjecture seems to produce nontrivial results in both the bulk and boundary increases the plausibility of the correctness of the original $E_p = E_W$ conjecture.

The super-Bayesian inequality in the context of the conditional entanglement of purification is another example of an inequality that is only true holographically (i.e. not for an arbitrary quantum state), much like the strong superadditivity in [96]. As such, it can be used as another discriminator for which quantum states are permitted to have holographic duals.

It has recently been proposed by [62] that the entanglement of purification can be calculated in 2D CFTs. If this is successful, it would be an interesting future direction to see if that method can prove the $E_p = E_W$ conjecture or, beyond that, any further generalization. Furthermore, perhaps E_W surfaces anchored to boundary-anchored HRT surfaces [66] that

[5]For a recent holographic analysis of this, see for example [13].

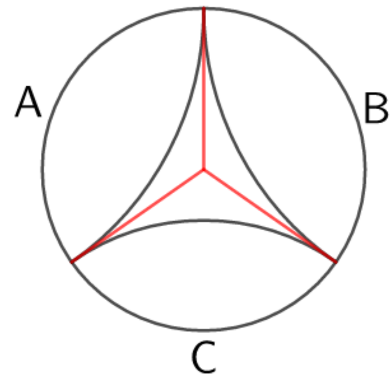


Figure 6.5: Counterexample showing that tripartite E_p and E_W are not dual to each other. Minimality of the RT surfaces imply that $E_W \geq \frac{1}{3}\left(S(A)+S(B)+S(C)\right)$, with the inequality being generically strict. Note, however, that a regulator is needed to makes sense of this statement since otherwise both sides of the inequality are divergent.

probe behind the event horizon of black holes formed from collapse [65] have areas which can be calculated both holographically and directly in the boundary field theory, providing a nontrivial check of the smoothness of the region behind the black hole horizon in black holes formed by collapse.

Finally, the usage of the conditional and multipartite entanglements of purification to partition bulk minimal surfaces is of great use in assigning Hilbert space factors to different subregions of the bulk, and will be something that will be used to some effect in building tensor networks via entanglement distillation [14].

Appendix A

Bondi Mass and BMS Transformations

In this appendix, based on sections of [23], we calculate transformations of the Bondi mass aspect and the angular momentum aspect under BMS transformations to extract frame invariant information. We show that there is no information in the Bondi mass aspect other than that contained in the Bondi 4-momentum.

A.1 Bondi Frames

Given an asymptotically flat space-time, one can always write the metric near $\mathscr{I}^+$ in the general form [49]

$$ds^2 = -Ue^{2\beta}du^2 - 2e^{2\beta}dudr + r^2\gamma_{AB}(d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du) \tag{A.1}$$

where u is the retarded time, r is a radial cordinate and A, B runs over$1, 2$, and Θ^A are coordinates on the topological two-spheres which are cross-sections of $\mathscr{I}^+$. The gauge freedom in the metric has been partially fixed by choosing

$$\begin{aligned} g_{rr} &= 0 \\ g_{rA} &= 0 \\ \partial_r\gamma &= 0 \end{aligned} \tag{A.2}$$

The functions in Eq. (A.1) have the following expansions in r

$$\begin{aligned}
\beta &= \frac{\beta_0}{r} + O(r^{-2}) \\
U &= 1 - \frac{2m}{r} + O(r^{-2}) \\
\gamma_{AB} &= h_{AB} + \frac{1}{r}C_{AB} + \frac{1}{r^2}D_{AB} + O(r^{-3}) \\
\mathcal{U}^A &= \frac{1}{r^2}U^A + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A(C_{BC}C^{BC}) + \frac{1}{2}C^{AB}D^C C_{BC}\right] + O(r^{-4}) \qquad \text{(A.3)}
\end{aligned}$$

The functions on the right hand sides of the above expansion are all functions of (u, θ^A). Furthermore h_{AB} is required to be the usual metric on the 2-sphere, and D_A is the covariant derivative with respect to h_{AB}. For use below we also note that $\beta_0 = 0$ and $U_A = -\frac{1}{2}D^B C_{AB}$, which follows from Einstein's Field Equations (EFE). Eqs. (A.1), (A.2), and (A.3) together with the conditions imposed by Einstein's equations constitute a **Bondi frame**. The tensor C_{AB} is sometimes called the shear tensor, as it measures deformations from the 2-sphere metric. C_{AB} is traceless and satisfies $\partial_u C_{AB} = N_{AB}$ where N_{AB} is the Bondi news tensor and measures gravitational radiation reaching $\mathscr{I}^+$.

The quantities m and N^A are respectively the Bondi mass aspect and angular momentum aspect, whose interpretation will be discussed in upcoming sections. Their evolution equations follow from EFE [49]:

$$\dot{m} = -4\pi\hat{T}_{uu} - \frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}D_A D_B N^{AB} = -4\pi\mathcal{T} + \frac{1}{4}D_A D_B N^{AB}, \text{and} \qquad \text{(A.4)}$$

$$\begin{aligned}
\dot{N}_A = &- 8\pi\hat{T}_{uA} + \pi D_A \partial_u \hat{T}_{rr} + D_A m + \frac{1}{4}D_B D_A D_C C^{BC} - \frac{1}{4}D_B D^B D^C C_{CA} \\
&+ \frac{1}{4}D_B(N^{BC}C_{CA}) + \frac{1}{2}D_B N^{BC}C_{CA}, \qquad \text{(A.5)}
\end{aligned}$$

where $\mathcal{T} = \hat{T}_{uu} + \frac{1}{32\pi}N_{AB}N^{AB}$ is called the effective stress-energy tensor, and includes contributions from both matter (first term) and gravitational radiation (second term). The boundary stress-energy tensor $\hat{T}$ is defined as the following limit of the bulk stress-energy tensor

$$\hat{T}_{\mu\nu}(u, \theta^A) = \lim_{r\to\infty} r^2 T_{\mu\nu}(u, r, \theta^A). \qquad \text{(A.6)}$$

The **BMS group** is the subgroup of diffeomorphisms that preserves the metric structure at $\mathscr{I}^+$. Besides containing the Poincare group as a subgroup (though, as we shall see, the Lorentz subgroup of BMS is far from unique), the BMS group also contains the so-called supertranslations, which are angle-dependent translations of retarded time: $u \to u + \alpha(\theta^A)$.

An arbitrary infinitesimal BMS transformations of $\mathscr{I}^+$ can be written as:

$$\vec{\xi} = [\alpha(\theta^A) + \frac{1}{2}uD_A Y^A(\theta^B)]\partial_u + Y^A(\theta_B)\partial_A \tag{A.7}$$

where Y^A are the conformal killing fields of the 2-sphere. That is, they can be written in the form

$$Y^A = D^A\chi_1 + \varepsilon^{AB}D_B\chi_2 \tag{A.8}$$

where $\chi_{1,2}$ are $l = 1$ spherical harmonics i.e.

$$D^2\chi_{1,2} = -2\chi_{1,2}. \tag{A.9}$$

One can extend (A.7) off of $\mathscr{I}^+$ by requiring that it take Bondi frames to Bondi frames. Note that this entails an appropriate rescaling of r. The result is [49]

$$\begin{aligned}\vec{\xi} &= f\partial_u + \left[Y^A - \frac{1}{r}D^A f + \frac{1}{2r^2}C^{AB}D_B f + O(r^{-3})\right]\partial_A \\ &- \left[\frac{1}{2}rD_A Y^A - \frac{1}{2}D^2 f - \frac{1}{2r}U^A D_A f + \frac{1}{4r}D_A(D_B f C^{AB}) + O(r^{-2})\right]\partial_r\end{aligned} \tag{A.10}$$

where $f(u, \theta^A) = \alpha(\theta^A) + \frac{1}{2}uD_A Y^A(\theta^B)$.

A.2 Bondi Mass Aspect

The Bondi mass aspect m appearing in the metric expansion [see Eqs. (A.1) and (A.3)] is intimately related to the mass inside the spacetime. This can be seen by looking, for instance, at the Schwarszchild solution, for which m is the Schwarszchild mass. In general though, m will not be constant on a cut, and it is its average value, known as the Bondi mass M, that is interpreted as the mass inside the spacetime. As expected, M decreases whenever radiation reaches $\mathscr{I}^+$. In this section we look at its transformation properties under the BMS group in increasingly more general spacetimes. We will make use of the general formula for the change of the Bondi mass aspect under a BMS transformation, which follows from its definition and Eq. (A.10):

$$\delta m = f\dot{m} + \frac{1}{4}N^{AB}D_A D_B f + \frac{1}{2}D_A f D_B N^{AB} + \frac{3}{2}m\psi + Y^A D_A m + \frac{1}{8}C^{AB}D_A D_B\psi, \tag{A.11}$$

where $\psi = D_A Y^A$.

Minkowski space

We begin our analysis in the simplest possible context: Minkowski space, which has $T_{\mu\nu} = 0$ and $N_{AB} = 0$ everywhere. Altogether, with the evolution equation (A.4), this implies $\dot{m} = 0$. In its usual Minkowski coordinates, we also have $m = 0$ and $C_{AB} = 0$. However, under a coordinate transformation one may change C_{AB}. Nonetheless, as we shall see, the mass aspect will remain invariant. To be able to show this, we use the fact that in vacuum EFE impose a constraint on C_{AB}, namely that it can be written as:

$$C_{AB} = \left(D_A D_B - \frac{1}{2} h_{AB} D^2 \right) c, \tag{A.12}$$

for some u-independent function c on the 2-sphere.

Recall that $\psi \equiv D_A Y^A$, where Y^A is given by Eq. (A.8). For a boost $Y^A = D^A \chi$, while for a rotation $Y^A = \epsilon^{AB} D_B \chi$, where in both cases χ is an $\ell = 0$ and $\ell = 1$ spherical harmonic. Non-vanishing ψ only occurs for $Y^A = D^A \chi$ with χ and $\ell = 1$ harmonic.

From Eq. (A.11), dropping out the terms that obviously vanish in Minkowski with $m = 0$,we are left with:

$$\delta m = \frac{1}{8} C^{AB} D_A D_B \psi = \frac{1}{8} \left(D^A D^B - \frac{1}{2} h^{AB} D^2 \right) c \;\; D_A D_B \psi. \tag{A.13}$$

This vanishes for $\psi = 0$, so letting $\psi = D_A Y^A = D^2 \chi = -2\chi$ for χ an $\ell = 1$ harmonic, we get:

$$\delta m = -\frac{1}{4} \left(D^A D^B - \frac{1}{2} h^{AB} D^2 \right) c \;\; D_A D_B \chi = \frac{1}{4} \chi h_{AB} \left(D^A D^B - \frac{1}{2} h^{AB} D^2 \right) c = 0. \tag{A.14}$$

Above, we have used that for an $\ell = 1$ spherical harmonic $D_A D_B \chi = -h_{AB} \chi$ (and also $D^2 = h_{AB} D^A D^B$ and $h_{AB} h^{AB} = 2$).

If we had found $\delta M \neq 0$ in Minkowski, this would imply that M would not be a good observable.

Stationary Region

Given that $\delta m = 0$ in Minkowski, we can now turn to showing that a well-defined notion of Bondi mass exists in stationary regions of $\mathscr{I}^+$. Note that this is a stronger condition than non-radiative ($N_{AB} = \hat{T} = 0$, or equivalently, assuming the null energy condition, $\mathcal{T} = 0$). Since C_{AB} and m are u-independent, we still have $\dot{m} = 0$ and $C_{AB} = (2 D_A D_B - h_{AB} D^2) c$, for some function $c(\theta, \phi)$ on the 2-sphere.

	Minkowski	Stationary
General Frame	$0=\hat{T}_{uu}=N_{AB}=m=\dot{N}_A$ and $C_{AB}=(2D_AD_B-h_{AB}D^2)c$	$\mathcal{T}=0$, and $C_{AB}=(2D_AD_B-h_{AB}D^2)c$
Rest Frame	Same as above (all Bondi frames are rest frames)	Above conditions, plus $\dot{N}_A=0$ and $m=m_0$
Canonical Frame	Above conditions, plus $C_{AB}=0$, and N_A magnetic	Above conditions, plus $C_{AB}=0$, and N_A magnetic

Table A.1: Contraints on quantities appearing in the Bondi metric for different spacetimes and in different frames.

Definition of the invariant mass

In a stationary region, it is always possible to find a frame in which $\dot{N}_A=0$. [49]. This is called the "rest frame" for reasons that shall become clear. We define the Bondi mass as follows: go to a frame with $\dot{N}_A=0$. We will show that in this frame the Bondi mass aspect is constant on the 2-sphere: $m=m_0$. This constant is identified as the invariant Bondi mass.

There are two things we need to show for this construction to be well defined. First, that for $\dot{N}_A=0$, the Bondi mass aspect is indeed constant; and that this constant is unique. To show m is constant we use the evolution equation for N_A [Eq. (A.5)], which for a region with no news or matter reads:

$$\dot{N}_A = D_A m + \frac{1}{4} D_B D_A D_C C^{BC} - \frac{1}{4} D^2 D^C C_{AC} \tag{A.15}$$

We may now proceed in one of two ways to show that the above equation with $\dot{N}_A=0$ implies m is constant in a stationary region. The easiest is to supertranslate to the frame in which $C_{AB}=0$. The second method, which involves a little more algebra to show directly that $D_BD_AD_CC^{BC}-D^2D^CC_{AC}=0$ in a stationary region, thus implying the desired result, is done in section A.3.

Note that if $C_{AB}=0$, then from the above evolution equation $\dot{N}_A=0$ implies $m=m_0$, a constant. Since we can write

$$C_{AB} = \left(2D_AD_B - h_{AB}D^2\right)c, \tag{A.16}$$

for some function c on the sphere, any vacuum C_{AB} can be made zero with a supertranslation, which is defined by a function $\alpha(\theta^A)$ and acts on C_{AB} as

$$\delta C_{AB} = \left(h_{AB}D^2 - 2D_AD_B\right)\alpha. \tag{A.17}$$

By picking $\alpha = c$, we make $C_{AB} = 0$. Importantly, a supertranslation changes neither m nor $\dot{N}_A$ as follows from δm and δN_A equations[49] for a stationary region of $\mathscr{I}^+$:

$$\delta m = \frac{3}{2} m\psi + Y^A D_A m, \tag{A.18}$$

$$\begin{aligned} \delta N_A = f\dot{N}_A + 3mD_A f + \frac{1}{4} C_{AB} D^B D^2 f - \frac{3}{4} D_B f (D^B D_C C^C{}_A - D_A D_C C^{BC}) \\ + \frac{3}{8} D_A (C^{BC} D_B D_C f) + \frac{1}{4} (2D_A D_B f - h_{AB} D^2 f) D_C C^{BC} \\ + \mathcal{L}_Y N_A + \psi N_A - \frac{1}{2} D_B (\psi D^{BA}) - \frac{1}{32} D_A \psi C_{BC} C^{BC}. \end{aligned} \tag{A.19}$$

To show uniqueness, we look at how an arbitrary BMS transformation acts on m. Could it be that $m = m_0$ in a frame, but m_0' in a different frame? Note that by Eq.(A.15), if these frames existed, both would have $\dot{N}_A = 0$. Also, there would be a transformation taking $m = m_0$ into $m = m_0'$. However, under a BMS transformation, m transforms as:

$$m \to m(\phi(\theta^A))\omega^{-3}, \tag{A.20}$$

where ϕ is a map of the sphere onto itself and $\omega = 1$ for rotations and $\omega(n^i) = \cosh(\eta) + \sinh(\eta)(n^i m_i)$ for a boost of rapidity η in the m^i direction. Here n^i and m^i are unit vectors. [1]

Note that since $\dot{m} = 0$ in the region of interest, supertranslations do not change m. From the form of ω together with Eq.(A.20), we see that if m starts as a constant, rotations will not change it, while any non-trivial ($\eta \neq 0$) boost will make it non-constant. Thus, m_0 is unique and the Bondi mass is well-defined in stationary regions.

The argument in this subsection shows that the Bondi mass M is the only information contained in the Bondi mass aspect $m(\theta^A)$ of a stationary region. Next, we show how this information can be obtained (in any frame) from the $\ell \leq 1$ modes of m.

The Bondi 4-Momentum

We now define a full 4-vector Bondi momentum p^μ, which in the rest frame ($\dot{N}_A = 0$) is given by $(m_0, \vec{0})$, and transforms as a special-relativistic 4-vector does. Under a boost with rapidity η along the first spacial direction, it goes to $p'^\mu = (m_0 \cosh(\eta), m_0 \sinh(\eta), 0, 0)$. Note that, as suggested by its name, in the "rest frame", the spatial momentum is zero.

We can now identify the components of this vector with $\ell = 0$ and $\ell = 1$ modes of m. We denote by $m_{\ell,n}$ the (ℓ, n) mode of m :

[1] Regarding indices, we follow the convention that Greek letters denote spacetime indices (running from 0 to 4), lower case Latin letters denote spatial indices (running from 1 to 3), while capital Latin letter denote 2-sphere indices (running over $\{1, 2\}$).

$$m_{\ell,n} = \int d^2\Omega m Y^*_{\ell,n}, \tag{A.21}$$

where $Y_{\ell,n}$ are spherical harmonics, $d^2\Omega$ is the area element of the unit 2-sphere, $*$ denotes complex conjugation. Starting with $m = m_0$ and boosting into m' given by Eq.(A.20), we can explicitly calculate the $\ell = 0$ and $\ell = 1$ modes of m' :

$$\begin{aligned}
m'_{0,0} &= \int d^2\Omega \frac{m_0}{\omega^3} Y_{0,0} = m_0 Y_{0,0} \int d^2\Omega \frac{1}{(\cosh(\eta) + \sinh(\eta)\cos\theta)^3} = 2\sqrt{\pi}\cosh(\eta) m_0, \\
m'_{1,-1} &= \int d^2\Omega \frac{m_0}{\omega^3} Y^*_{1,-1} = 0, \\
m'_{1,1} &= \int d^2\Omega \frac{m_0}{\omega^3} Y^*_{1,1} = 0, \\
m'_{1,0} &= \int d^2\Omega \frac{m_0}{\omega^3} Y^*_{1,0} = m_0 \int d^2\Omega \frac{Y_{1,0}}{(\cosh(\eta) + \sinh(\eta)\cos\theta)^3} = 2\sqrt{3\pi}\sinh(\eta) m_0.
\end{aligned} \tag{A.22}$$

where for simplicity, but without loss of generality, we chose to boost along the $\theta = 0$ direction. This implies

$$p^\mu = \frac{1}{2\sqrt{\pi}}(m_{0,0}, p^i) \tag{A.23}$$

$$p_i p^i = \frac{1}{3}(|m_{1,-1}|^2 + |m_{1,0}|^2 + |m_{1,1}|^2). \tag{A.24}$$

Thus, $p^2 = -m_0^2$ is a BMS invariant (as long as supertranslations don't take you out of the $N_{AB} = 0$ region). By calculating the $\ell \leq 1$ components of $m'_{\ell,n}$ under boosts in a general direction, it's possible to determine the exact form of all momentum components:

$$p^\mu = \frac{1}{2\sqrt{\pi}}\left(m_{0,0}, \frac{1}{\sqrt{6}}(m_{1,1} - m_{1,-1}), \frac{i}{\sqrt{6}}(m_{1,1} + m_{1,-1}), \frac{1}{\sqrt{3}} m_{1,0}\right) \tag{A.25}$$

These can be identified as proportional to the *real* spherical harmonic components of m. The Bondi 4-momentum, as written here, is well defined, transforms appropriately, can be measured in any Bondi frame, and has the same magnitude in all of them.

One might worry that the $m_{\ell,n}$ mode with $\ell \geq 2$ will not in general be zero:

$$m'_{\ell,n} = \int d^2\Omega \frac{m_0}{\omega^3} Y^*_{\ell,n} \tag{A.26}$$

Nonetheless, in a stationary region, all these higher modes can be made to vanish; thus, in such a region, they cannot contain new information. It is possible however, they redundantly contain some of the information in p^μ.

General Region

In a general region, which can contain Bondi News, a supertranslation will change the Bondi mass, according to Equation (A.11):

$$\delta m = f\dot{m} + \frac{1}{4}N^{AB}D_A D_B f + \frac{1}{2}D_A f D_B N^{AB} = -4\pi f\mathcal{T} + \frac{1}{4}D_A D_B(fN^{AB}), \tag{A.27}$$

where we have used the Eq. (A.4).

This change can be identified as being due to the flux of energy reaching $\mathscr{I}^+$. To see this, let's look at the change in the Bondi mass:

$$\delta M = \frac{1}{4\pi}\int d^2\Omega\delta m = -\int d^2\Omega f\mathcal{T}. \tag{A.28}$$

We can make sense of this physically: the supertranslation is taking a cut at some $u = u_0$, constant, and taking it into another cut $u = u_0 + f$. The change in the mass is proportional to the flux arriving between these two cuts. (The transformation is infinitesimal so that $\mathcal{T}$ us assumed independent of u in the region between the cuts.)

One can similarly write the transformation law for all $m_{\ell,n}$ under supertranslations:

$$\delta m_{\ell,n} = \int d^2\Omega\delta m Y^*_{\ell,n} = \int d^2\Omega\left(-4\pi f\tau + \frac{1}{4}D_A D_B(fN^{AB})\right)Y^*_{\ell,n} \tag{A.29}$$

A.3 Alternative proof that Bondi Mass is well defined

We now prove the following lemma to show that $m = m_0$ in a stationary region in a Bondi frame with $\dot{N}_A = 0$.

Lemma: $D_B D_A D_C C^{BC} - D^2 D^C C_{AC} = 0$ in a region of $\mathscr{I}^+$ devoid of matter and radiation.

Proof: First note a useful fact about R_{ABCD} for a 2-sphere that will be used throughout:

$$R_{ABCD} = h_{AC}h_{BD} - h_{AD}h_{BC} \tag{A.30}$$

We therefore have

$$\begin{aligned} D_A D^C C_{BC} - D^C D_A C_{BC} &= R^D{}_B{}^C{}_A C_{DC} + R^D{}_C{}^C{}_A C_{BD} \\ &= R^D{}_B{}^C{}_A C_{DC} - R^D{}_A C_{BD} \\ &= -2C_{AB} \end{aligned} \tag{A.31}$$

where we have used the fact that C_{AB} is traceless, i.e. $h^{AB}C_{AB} = 0$.

Exchanging indices $A \leftrightarrow B$ in Eq.(A.31) yields

$$D_B D^C C_{AC} - D^C D_B C_{AC} = -2C_{AB} \tag{A.32}$$

since C_{AB} is symmetric.

From this we get the result

$$\begin{aligned} D_B D_A D_C C^{BC} - D_B D^B D^C C_{CA} = D^B D^C D_A C_{BC} - 2D^B C_{AB} - D^B D^C D_B C_{AC} + 2D^B C_{AB} \\ = 2D^B D^C D_{[A} C_{B]C} \end{aligned} \tag{A.33}$$

In order to show Eq.(A.33) vanishes, we first recall that in a stationary region

$$C_{AB} = (D_A D_B - \frac{1}{2} h_{AB} D^2) c \tag{A.34}$$

for any smooth function $c(\theta, \varphi)$ on the 2-sphere.

We then compute

$$\begin{aligned} 2D_{[A} C_{B]C} &= D_A D_B D_C c - D_B D_A D_C c - \frac{1}{2} h_{BC} D_A D^2 c + \frac{1}{2} h_{AC} D_B D^2 c \\ &= R_{ABC}{}^D D_D c - \frac{1}{2} h_{BC} D_A D^2 c + \frac{1}{2} h_{AC} D_B D^2 c \end{aligned} \tag{A.35}$$

and inserting this into Eq.(A.33) yields

$$\begin{aligned} 2D^B D^C D_{[A} C_{B]C} &= D^B D^C (R_{ABC}{}^D D_D c) - \frac{1}{2} D^B (D_B D_A - D_A D_B) D^2 c \\ &= D^B D^C (R_{ABC}{}^D D_D c) \end{aligned} \tag{A.36}$$

where in the second line we have used that D_A commutes on scalar fields.

Furthermore from Eq.(A.30) it follows that

$$R_{ABC}{}^D = h_{AC} \delta_B{}^D - h_{BC} \delta_A{}^D \tag{A.37}$$

which, when inserted into Eq.(A.36), results in

$$D^B D^C (R_{ABC}{}^D D_D c) = D^D D_A D_D c - D^2 D_A c = 0 \tag{A.38}$$

where we have again used that D_A commutes on scalar fields.

Thus we have

$$D^B D^C D_{[A} C_{B]C} = 0 \tag{A.39}$$

which immediately implies the desired result. □

This lemma, together with Eq. (A.15), establishes that $m = m_0$ in stationary regions of $\mathscr{I}^+$ in the frame with $\dot{N}_A = 0$.

Appendix B

KRS Bound

Ultimately the equivalence principle is a hypothesis, supported by a certain amount of evidence. Indeed, Strominger [92] (building on earlier work of Ashtekar [7] and others) has argued that the vacuum of asymptotically flat spacetimes, Minkowski space, is infinitely degenerate, i.e., that it corresponds to an infinite number of distinct quantum states labeled by the quantity C_{AB} in Eq. (2.39).

If these states could be distinguished by any observation, empty space would contain an infinite amount of information. This would constitute a violation of the equivalence principle in its usual, classical sense: in a basis where Minkowski-like states are labeled by C_{AB}, they can be naturally identified with a choice of coordinates, so coordinates would be measurable. Kapec, Raclariu, and Strominger [67] (KRS) recently proposed an entropy bound that contains an extra term (denoted $X^{\mathrm{K}RS}$ below), designed to account for this possibility.

A precise definition of the relevant entropy was not yet given in Ref. [67]. More importantly, no measurement protocol has been suggested for extracting the information contained in empty space. Such a measurement would rule out the equivalence principle experimentally. Conversely, absent experimental evidence to the contrary, we would argue that the equivalence principle should be retained: we should not consider Minkowski space written with different coordinate parameters C_{AB} to be physically distinct spacetimes.

In order to facilitate further study, we will summarize our understanding of the differences between our bounds and the KRS bound. We will offer a geometric interpretation of the extra term $X^{\mathrm{K}RS}$. We will explain why it appears in their derivation of an asymptotic bound but not in ours. We will also describe how the presence of this term conflicts with the equivalence principle.

KRS considered the asymptotic limit of bulk null hypersurfaces with approximately spherical cross-sections. This simplifies the approach to $\mathscr{I}^+$ in spherical Bondi coordinates, compared to our use of planar light-sheets. Unlike the planar null surfaces $H(u_p)$ used above, however, existing bulk entropy bounds become divergent and hence trivial in the asymptotic limit of spherical null surfaces. Hence they cannot be used as a starting point if one wishes to work with spherical cross-sections. A new subtraction method was proposed to cancel the divergence [67]. The KRS bound is

$$S_0^{\mathrm{KR}S}[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] + X^{\mathrm{KR}S}[\hat{\sigma}_2; C_{AB}(\infty)] \tag{B.1}$$

where $u_2(\Omega)$ defines the position of the cut.[1] A full definition of $S_0^{\mathrm{KR}S}$ was left to future work, but we will argue below that the choice is tightly constrained by coordinate invariance.

Let us first consider the r.h.s. of Eq. (B.1). The first term is given by

$$\Delta K[\hat{\sigma}_2] \equiv \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega \, du \, [u - u_2(\Omega)] \, \hat{\mathcal{T}} \; . \tag{B.2}$$

This is precisely the r.h.s. of our integrated Boundary GSL, Eq. (2.19):

$$\hat{S}_C[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] \; . \tag{B.3}$$

However the r.h.s. of the KRS bound contains the extra term

$$\begin{aligned} X^{\mathrm{KR}S} &\equiv -\frac{1}{8G\hbar} \int d^2\Omega \; D_A u_2(\Omega) \; D_B \bar{C}^{AB} && \text{(B.4)} \\ &= \frac{1}{8G\hbar} \int d^2\Omega \; u_2(\Omega) \; D_A D_B \bar{C}^{AB} && \text{(B.5)} \end{aligned}$$

where

$$\bar{C}^{AB} \equiv C^{AB}(\infty) - C_0^{AB} \; . \tag{B.6}$$

Here C_0^{AB} refers to a fiducial choice of Bondi coordinates (or of a "late-time vacuum" in the sense of Ref. [92]) at $u \to \infty$, whereas C^{AB} refers to the "actual" Bondi coordinates (or "late-time vacuum") that will be attained as $u \to \infty$. Because $D_A D_B \bar{C}^{AB}$ is a total derivative, its average on the cut $\hat{\sigma}_2$ vanishes, so unless it vanishes identically, it will have indefinite sign on the sphere. It also follows that $X^{\mathrm{KR}S} = 0$ if $u_2 = \text{const}$, so the extra term only contributes if the cut has nontrivial angular dependence in the chosen coordinates.

In the bulk, $D_A D_B \bar{C}^{AB}$ arises geometrically from a nonvanishing expansion of the null hypersurfaces at late times, which remains after the KRS regularization. Namely, the null expansion orthogonal to a surface of constant u, r in Minkowski space in the metric of Eq. (2.39) is

$$\theta[C_{AB}] = -\frac{1}{r} - \frac{1}{2r^2} D^A D^B C_{AB} \; , \tag{B.7}$$

so the difference between two choices C^{AB}, C_0^{AB} yields

$$\bar{\theta}(\Omega) = -\frac{1}{2r^2} D^A D^B \bar{C}_{AB} \; . \tag{B.8}$$

Substituting this result in Eq. (B.5), the term $X^{\mathrm{KR}S}$ can thus be understood as an extra area difference accumulated due to a nonzero regulated expansion $\bar{\theta}$ of the KRS null surface at late times.

[1] In the notation of Ref. [67], our $\hat{\sigma}_2$ is their Σ; our ΔK is $-A_F^{\Sigma}/4G\hbar$; and our $X^{\mathrm{KR}S}$ is $(-A_0^{\Sigma} + A_F^{\Sigma})/4G\hbar$.

The extra term $X^{\mathrm{K}RS}$ was motivated in Ref. [67] by covariance of their geometric construction under BMS transformations, so it is worth explaining its absence in Eq. (B.3) and our other bounds. KRS consider a coordinate sphere at fixed u, r at late times, and construct a null hypersurface orthogonal to it. BMS transformations act nontrivially by deforming the geometry of this coordinate sphere and changing its null expansion as a function of angle. This change propagates along the entire null hypersurface and leads to an extra area difference $X^{\mathrm{K}RS}$ as described above.

A bulk BMS supertranslation of a given late-time cross-section of the null plane $H(u_p)$ would yield a similar term. The null surface $\tilde{H}(u_p)$ orthogonal to the new cross-section would be neither a light-sheet nor a causal horizon, because the expansion at late times has the wrong sign on some generators. From this perspective, the KRS conjecture involves a modification of the nonexpansion condition of the covariant entropy bound, such that the permitted range of the (regulated) expansion depends on the late-time Bondi frame.

However, with our definition of $H(u_p)$, BMS supertranslations do not act in this way. We defined $H(u_p)$ not in terms of a given bulk cross-section, but as the boundary of the past of a point p on $\mathcal{I}^+$ [25]. BMS supertranslations can only move this point along the null geodesic generator on which it lies. The boundary of the past of any point on $\mathcal{I}^+$ has vanishing late-time expansion and is a causal horizon. Thus, supertranslations map the set of all $H(u_p)$ to itself. Therefore they have no effect when the limit as $u_p \to \infty$ is taken, and they leave no imprint in our asymptotic entropy bounds.

We now turn to the l.h.s. of Eq. (B.1). The indefinite sign of $D_A D_B C^{AB}(\infty)$ on the sphere constrains possible definitions of $S_0^{\mathrm{K}RS}$. It implies that $S_0^{\mathrm{K}RS}[\sigma_2]$ cannot be unique in Minkowski space, for any nonconstant cut σ_2. In particular, it is not possible for $S_0^{\mathrm{K}RS}$ to always vanish for arbitrary subregions of the boundary of Minkowski space regardless of the choice of coordinates.

To see this, choose asymptotic coordinates such that $\bar{C}_{AB} = \beta \tilde{C}_{AB}$ where $\tilde{C}_{AB}$ is nonvanishing and satisfies Eq. (2.41), and β is a constant. By Eq. (B.4), $X^{\mathrm{K}RS}$ is linear in β so it can be made negative and arbitrarily large in magnitude by an appropriate choice of β. This would violate the KRS bound so $S_0^{\mathrm{K}RS}[\sigma_2]$ must depend on $\bar{C}_{AB}$.

The above considerations also imply that $S_0^{\mathrm{K}RS}$ cannot be bounded from below by the Shannon entropy—not even approximately—in the case where classical Bondi news is present.

Indeed, KRS advocate that $S_0^{\mathrm{K}RS}$ should not be unique in Minkowski space. Rather it should contain a "soft term" that depends on C^{AB} in some way, so that the KRS bound is satisfied independently of the choice of the "reference vacuum" C_0^{AB}.

Here we note that the only definition consistent with the equivalence principle is

$$S_0^{\mathrm{K}RS} \equiv \hat{S}_C + X^{\mathrm{K}RS} \ , \tag{B.9}$$

where S_C has no dependence on C_{AB}. With this choice, the $X^{\mathrm{K}RS}$ terms would cancel, and thus, all dependence on C_{AB} would drop out. Then Eq. (B.1) would reduce to Eq. (B.3). With any inequivalent definition, the physical content of Eq. (B.1) would depend on a coordinate choice.

This is because $X^{\mathrm{K}RS}$ depends on the quantity $\bar{C}_{AB}$ defined in Eq. (B.6). We have argued in Sec. 2.3 that $C_{AB}(\infty)$ can be changed by changing the coordinate choice. Therefore, neither $C_{AB}(\infty)$ nor its difference from a fiducial value, $\bar{C}_{AB}$, can be observable, if the equivalence principle is valid. [Note that the fiducial value C_0^{AB} need *not* correspond to the value of C_{AB} at any cut on $\mathcal{I}^+$. If it did, $\bar{C}_{AB}$ could be measured, and it would originate with physical radiation whose information content satisfies Eq. (B.3).]

In particular, if $S_0^{\mathrm{K}RS}$ could be constructed entirely from observable quantities, then Eq. (B.1) could be used to constrain $\bar{C}_{AB}$, thus making it accessible to observation. This would be a problem: $\bar{C}_{AB}$ *must* remain unobservable by the equivalence principle, because it corresponds to a coordinate choice in Minkowski space.

In closing, we stress again that by the equivalence principle we mean the statement that empty Riemann-flat space contains no classical information. In Sec. 2.4 we showed that the bounds of Sec. 2.2 are consistent with this principle. In this appendix we have argued that the KRS bound is not consistent with it, except for a particular choice of definition of entropy under which it would reduce to Eq. (2.19). We make no claims about the compatibility of the KRS bound with any other formulation of the equivalence principle.

Appendix C

Single Graviton Wavepacket

In this appendix, we study the implications of asymptotic bounds in a quantum setting; we will find that in some cases they restrict the entropy more strongly than the equivalence principle did for classical waves.

We consider a classical probabilistic ensemble of single graviton wave packets, of characteristic wavelength λ in the u-direction. Like the classical gravitational wave of Sec. 2.4, the wave packets shall be roughly centered on $u = 0$, and delocalized on the sphere. This is a global quantum state, defined on all of $\mathcal{I}^+$.

Any such state is orthogonal to the vacuum. Here we shall take the global state ρ_g to be a mixed state with global von Neumann entropy of order unity:

$$\hat{S}_g = -\mathrm{tr}\,\rho_g \log \rho_g \sim O(1) \; . \tag{C.1}$$

For example, ρ_g could be an incoherent superposition of the graviton wavepacket in two different polarization states. Alice could encode a message about the weather in the choice of polarization, and Bob could decode this message if he is able to measure the polarization.

In the region occupied by the wave packet, we have

$$\begin{aligned} N_{AB}N^{AB} &\sim O\left(\frac{l_P^2}{\lambda^2}\right) \; , && \text{(C.2)} \\ \hat{\mathcal{T}} &\sim O\left(\frac{\hbar}{\lambda^2}\right) \; , && \text{(C.3)} \\ N_{AB} &\sim O\left(\frac{l_p}{\lambda}\right) \; , && \text{(C.4)} \end{aligned}$$

where expectation value brackets are left implicit. The gravitational memory created by the wavepacket is

$$\Delta C_{AB}^{\infty} = \int_{-\infty}^{\infty} N_{AB}\, du \approx \int_{-\lambda}^{\lambda} N_{AB}\, du \sim O(l_P) \; , \tag{C.5}$$

where

$$l_P \equiv \sqrt{G\hbar} \tag{C.6}$$

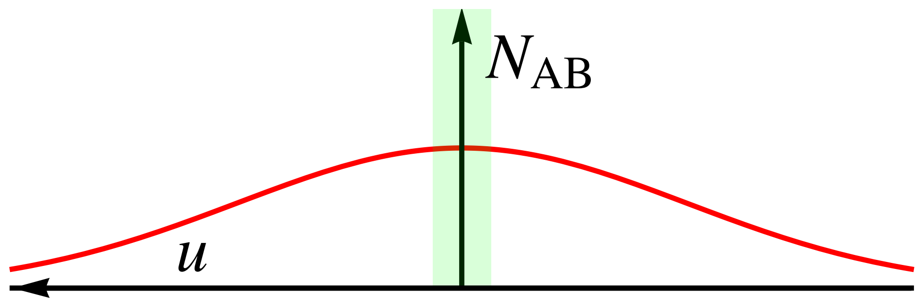


Figure C.1: A short observation (green shaded rectangle) cannot distinguished the reduced graviton state from the vacuum reduced to the same region. The graviton delivers no information to this observer.

is the Planck length. Note that the memory is independent of λ and so remains finite as λ is taken large.

Boundary Quantum Bousso Bound

The Boundary QBB, Eq. (2.16), bounds the entropy on finite portions of $\mathcal{I}^+$. This is particularly relevant to actual experiments. There are no experiments that started infinitely long ago and will complete an infinite time from now. When we measure something, we do it in finite time.

Hence, we will consider an experiment of finite duration of order T. It will be convenient to center this time interval near $u = 0$. Thus, we consider an observer who has access to the subregion

$$-T \lesssim u \lesssim T \tag{C.7}$$

of $\mathcal{I}^+$ (or to the subregion of the asymptotic region defined by the same range, in Bondi coordinates). It will not be important whether the cuts $\hat{\sigma}_1$, $\hat{\sigma}_2$ are at constant u.

All observables that can be measured by this observer can be computed from the reduced density operator

$$\rho_T \equiv \mathrm{tr}_{\not{T}}\, \rho_g \ . \tag{C.8}$$

We must also consider the global vacuum state, restricted to the observation interval:

$$\chi_T = \mathrm{tr}_{\not{T}}\, |0\rangle\langle 0| \ . \tag{C.9}$$

In this notation, the vacuum-subtracted entropy, Eq. (2.17), is written as

$$\hat{S}_C = -\mathrm{tr}_T\, \rho_T \log \rho_T + \mathrm{tr}_T\, \chi_T \log \chi_T \ . \tag{C.10}$$

The subscript T (or $\not{T}$) on the trace indicates that the trace is taken over the Hilbert space factor associated with the observation interval (or its complement).

Short Observation Regime We begin by considering the case where $\lambda \gg T$. In this regime, the observer has access to a region occupied by the graviton wavepacket, but much smaller than the wavepacket (Fig. C.1). The Boundary QBB implies

$$\hat{S}_C \lesssim O(\hat{\mathcal{T}} T^2/\hbar) \sim O(T^2/\lambda^2) \ , \tag{C.11}$$

so the upper bound vanishes quadratically with T/λ.

To understand this result, it is instructive to return to the bulk and consider the case of a scalar field wavepacket passing through $H(u_p)$. In this setting, the entropy can be computed explicitly; and the bound has been proven [35, 34]. A beautiful explanation of the vanishing of the information content was given by Casini [38], building on pioneering work of Marolf, Minic, and Ross [74].

To an observer with access to a finite or semi-infinite region, the vacuum (restricted to this region) is a noisy state. For example, in the simplest case of a semi-infinite region (Rindler space), the restricted vacuum is a thermal state. Further restrictions only make the fluctuations larger. This means that the global vacuum restricted to the interval $(-T, T)$ is a state in which thermal-like excitations with energy up to order $\hbar/T$ are unsuppressed. This energy is larger, by a huge factor λ^2/T^2, than the total energy of the graviton in this region. This is the physical origin of Eq. (C.11): because of thermal noise, states with and without the graviton wavepacket cannot be distinguished by an observer with access to a small subregion of the wavepacket. In short, the vacuum-subtracted entropy is a physical quantity that correctly captures how much information can be gained by a given observer.

We can also shift the observation interval so that it fails to overlap with the graviton. This is analogous to a case of classical Bondi news studied in Sec. 2.4, and it gives the same result: In this case it does not matter how long or short the observation is; if it does not overlap with the news, then upper bound vanishes.

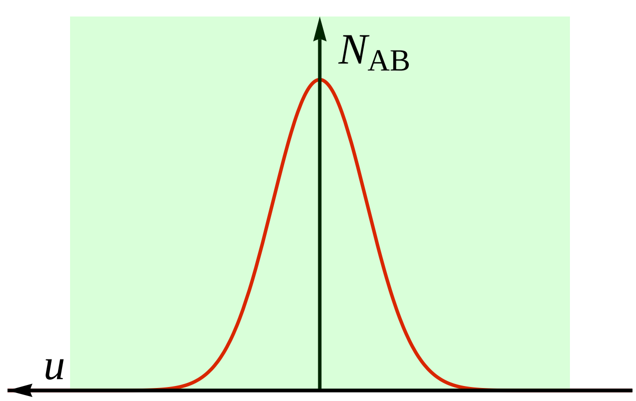


Figure C.2: A long observation (green shaded rectangle) can distinguish the reduced graviton state from the reduced vacuum. The graviton carries information to this observer.

Long Observation Regime Next, let us consider the case where the observer has access to a region that includes the whole wavepacket: $T \gg \lambda$ (Fig. C.2). In the long-observation regime, the experiment begins well before the graviton starts arriving, and ends well after. From Eq. (C.3) we see that the energy density $\hat{\mathcal{T}}$ scales as λ^{-2}. The Boundary QBB, Eq. (2.16), evaluates to

$$\hat{S}_C \lesssim O(\hat{\mathcal{T}} T\lambda/\hbar) \sim O\left(\frac{T}{\lambda}\right) , \tag{C.12}$$

as the integral has support only only on the central interval of size $O(\lambda)$ where $\hat{g} \sim O(T)$. Since we have $T \gg \lambda$, Eq. (C.12) is consistent with the ability of the observer to extract information from the graviton.

We may specify a "soft limit" of the long-observation regime as follows: Let $T = \alpha\lambda$, with $\alpha \gg 1$ fixed. Then we take λ to become as large as we like, while the experiment always lasts longer than the wavepacket. We note that the upper bound remains fixed in this limit, at $O(\alpha) \gg 1$. We can tighten the upper bound to $O(1)$ while remaining marginally within the long-observation regime by taking $\alpha \sim 1$.

We can gain further intuition by returning to the bulk and considering the same graviton as it crosses a planar light-sheet $H(u_p)$. It induces focussing as $d\theta/dw \sim O(G\hbar/(A\lambda^2))$, were A is the transverse area on which the wavepacket has support. Integrating twice along the light-rays and once transversally, we see that the area loss between the two ends of the wave packet is of order the Planck area for $\alpha \sim 1$. Thus, a single quantum induces loss of about a Planck area in planar light-sheets, independently of wavelength [27]. Hence the bound on its entropy is of order unity.

We will not try to compute the entropy of the graviton directly, but we expect it to be of order unity. To see this, let us again consider instead a scalar field wavepacket, for which the QBB has been proven [35, 34]. We understand the presence of nonzero entropy: the experiment can access the whole wavepacket, and the excitation can be distinguished reasonably well from the thermal noise that pollutes any finite-duration measurements [74, 38]. Thus as $\alpha \sim 1$, the bound becomes approximately saturated at the order-of-magnitude level.

Boundary Generalized Second Law

Finally, let us consider an observer with access to a semi-infinite region above some cut $\hat{\sigma}_2$ of $\mathcal{I}^+$. The bound that applies to this case is the integrated Boundary GSL, Eq. (2.19):

$$\hat{S}_C[\hat{\sigma}_2] \leq \Delta K[\hat{\sigma}_2] , \tag{C.13}$$

where

$$\Delta K[\hat{\sigma}_2] \equiv \frac{2\pi}{\hbar} \int_{\hat{\sigma}_2}^{\infty} d^2\Omega \, du \, [u - u_2(\Omega)] \, \hat{\mathcal{T}} \tag{C.14}$$

is the modular Hamiltonian.

We stressed earlier that all real experiments are finite. Nevertheless, the above bound is a useful approximation for long but finite observations: first specify the global state, which must obey fall-off conditions [41] on the news. Then restrict to an interval (u_2, T) such that T lies far inside the future region with essentially no news, and consider the QBB for this interval. Since the slope of $\hat{g}$ is unity near the lower end of the interval, and since $\hat{S}_C$ will no longer depend on T in this regime, the Boundary QBB reduces to Eq. (C.13).

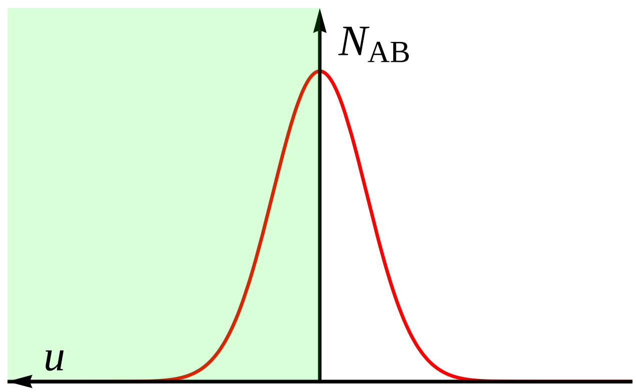


Figure C.3: A graviton conveys $O(1)$ information as long as it has appreciable support in the region of observation.

Let us apply Eq. (C.13) to a graviton wavepacket with support in the region $(-\lambda, \lambda)$. First suppose that the cut $\hat{\sigma}_2$ lies, say, around the center of the wavepacket, as depicted in Fig. C.3. By the previous paragraph, the results will be the same as for the QBB in the regime $\alpha \sim 1$: the asymptotic geometry can be distinguished from Minkowski space, and the upper bound will be of order unity. On the other hand, if we shift the wavepacket so as to lie entirely prior to $\hat{\sigma}_2$, then the upper bound vanishes.

We can also consider the differential version of the Boundary GSL, which can be written as

$$-\frac{1}{\delta\Omega}\frac{d}{du}\hat{S}_C[\hat{\sigma}_2; \Omega] \leq \frac{2\pi}{\hbar}\int_{\hat{\sigma}_2}^{\infty} du\ \hat{\mathcal{T}}\ . \tag{C.15}$$

This vanishes if the news has no support in the region above the cut $\hat{\sigma}_2$. Thus, for the case of news that arrives entirely prior to $\hat{\sigma}$, the upper bounds on the entropy, and on its variation under deformations of $\hat{\sigma}$, both vanish. This is the same behavior we encountered for the classical case in Sec. 2.4.

In the case where a graviton wavepacket lies partially or entirely above the cut (Fig. C.3), we see that the derivative of $\hat{S}_C$ is bounded by the energy of the wavepacket. This is nonzero for any finite λ. We note that the upper bound depends on the energy, not on the gravitational memory created by the wavepacket. Therefore, the upper bound on the derivative of $\hat{S}_C$ vanishes in the soft limit as λ becomes large, even though ΔC_{AB} remains fixed in this limit. Thus, the differential Boundary GSL implies that the variation in entropy of the region above a cut $\hat{\sigma}$, under fixed length deformations of $\hat{\sigma}$, is insensitive to the addition of gravitons of much greater wavelength.

Appendix D

Calculation of $\langle Q^2 \rangle$ and $\langle M^2 \rangle$

In this appendix we describe in more detail the calculations of $\langle Q^2 \rangle$ described in section 3.2 and of $\langle M^2 \rangle$ described in section 3.3.

For the QED calculation, we start with eq.(3.17). Inserting the expression for the current 2-point function, Eq. (3.5), and evaluating the $d^3\vec{x}$ integral, we get

$$\langle Q^2 \rangle = \kappa \int dx^0 d\Delta^0 4\pi^2 d\Delta \frac{(\Delta^2 + (\Delta^0)^2)\,\mathrm{Vol}(r_1(t), r_2(t), \Delta)}{\left(\Delta^2 - (\Delta^0)^2\right)^2} w(x^0) w(x^0 - \Delta^0), \tag{D.1}$$

Note that here $\Delta = |\vec{\Delta}|$, whereas in the main text Δ denoted a four-vector.

The radii, as functions of time, are specified by

$$\begin{aligned} r_1(t) &= r_B + \alpha(t - t_B), \\ r_2(t) &= r_B + \alpha(t - \Delta^0 - t_B), \end{aligned} \tag{D.2}$$

$w(t)$ is given in Eq.(3.13) and $\mathrm{Vol}(r_1, r_2, \Delta)$ is the volume of the intersection of two spheres of radii r_1 and r_2 whose centers are separated a distance Δ. Explicitly, the volume formula is

$$\mathrm{Vol}(r_1, r_2, \Delta) = \frac{\pi(-\Delta + r_1 + r_2)^2\left(\Delta^2 - 3\left({r_1}^2 + {r_2}^2\right) + 2\Delta(r_1 + r_2) + 6 r_1 r_2\right)}{12\Delta}, \tag{D.3}$$

for $|r_1 - r_2| \le \Delta \le r_1 + r_2$. For $\Delta > r_1 + r_2$, the spheres do not intersect, and so $\mathrm{Vol}(r_1, r_2, \Delta) = 0$. For $\Delta < |r_1 - r_2|$, one ball is inside the other and so $\mathrm{Vol}(r_1, r_2, \Delta) = \frac{4}{3}\pi \min(r_1, r_2)^3$. Evaluating the Δ integral in Eq. (D.1), we get

$$\begin{aligned} \langle Q^2 \rangle = \frac{16\pi^6\kappa}{15} \int dx^0 d\Delta^0 & \frac{{r_1}^3 {r_2}^3 \left(-5(\Delta^0)^4 + 2(\Delta^0)^2\left({r_1}^2 + {r_2}^2\right) + 3\left({r_1}^2 - {r_2}^2\right)^2\right)}{(\Delta^0 + r_1 - r_2)^3 (\Delta^0 + r_1 + r_2)^3 (-\Delta^0 + r_1 - r_2)^3 (-\Delta^0 + r_1 + r_2)^3} \\ & \times \frac{\delta t^2/\pi^2}{\left(\delta t^2 + {x^0}^2\right)\left(\delta t^2 + (x^0 - (\Delta^0)^2)\right)}. \end{aligned} \tag{D.4}$$

We now choose contours to evaluate, in turn, the $\varDelta^0$ and the x^0 integrals. Keep in mind that at this step the expressions for r_1 and r_2, Eq.(D.2), need to be explicitly inserted. Seen as a function on the complex $\varDelta^0$ plane, the integrand in Eq. (D.4) has four branch points, all on the real axis, and two simple poles, at $\varDelta^0 = x^0 \pm i\delta t$. We choose a contour that goes along the real axis, with infinitesimal deformations around the branch points to avoid them, and then close along a semi-circle on the upper half-plane (See Figure D.1). This contour picks up a residue at $\varDelta^0 = x^0 + i\delta t$, thus yielding

$$\langle Q^2 \rangle = \int dx^0 \frac{-\pi^4 \delta t \kappa \left(\frac{8((\alpha-1)r_B + i\delta t\alpha)((\alpha-1)r_B - \alpha x^0)\left(-\left(\alpha^2-1\right)\delta t^2 - 2i\delta t(x^0-(\alpha-1)\alpha r_B)+(\alpha-1)\left(2(\alpha-1)r_B^2 - 2\alpha r_B x^0 + (\alpha+1)(x^0)^2\right)\right)}{(\alpha-1)(\alpha+1)(\delta t(\alpha+1) - i(\alpha-1)(2r_B - x^0))(\delta t(\alpha-1) - i(2(\alpha-1)r_B - (\alpha+1)x^0))} \right)}{12\pi^3(\delta t - ix^0)^3(\delta t + ix^0)} - \frac{(\delta t - ix^0)^2 \log\left(\frac{\left(\delta t^2 + (-2(\alpha-1)r_B + \alpha(x^0 - i\delta t))^2 - 2i\delta t x^0 - (x^0)^2\right)^2}{\left(\alpha^2-1\right)^2 (x^0 + i\delta t)^4} \right)}{12\pi^3(\delta t - ix^0)^3(\delta t + ix^0)} \tag{D.5}$$

Looking at the integrand above as a function of x^0 on the complex plane we see that the branch points, in the limit of interest ($\alpha \to 1_-$), do not lie above the real line. Thus, the same contour prescription can be used to evaluate the x^0 integral, which now picks up a residue only at the simple pole at $x^0 = i\delta t$. Doing so, and using Eq.(3.15) to replace δt to δu, gives

$$\kappa\pi^2 \frac{-\alpha^2\left(\alpha^2-2\right)\delta u^4 + 8(\alpha-1)^4 \delta u^2 {r_B}^2 - \left(1-\alpha^2\right)\delta u^2\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right)\log\left(\frac{\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right)^2}{\left(\alpha^2-1\right)^2 \delta u^4}\right) + 16(\alpha-1)^8 {r_B}^4}{12(\alpha-1)(\alpha+1)\delta u^2\left(\delta u^2 + 4(\alpha-1)^4 {r_B}^2\right)}. \tag{D.6}$$

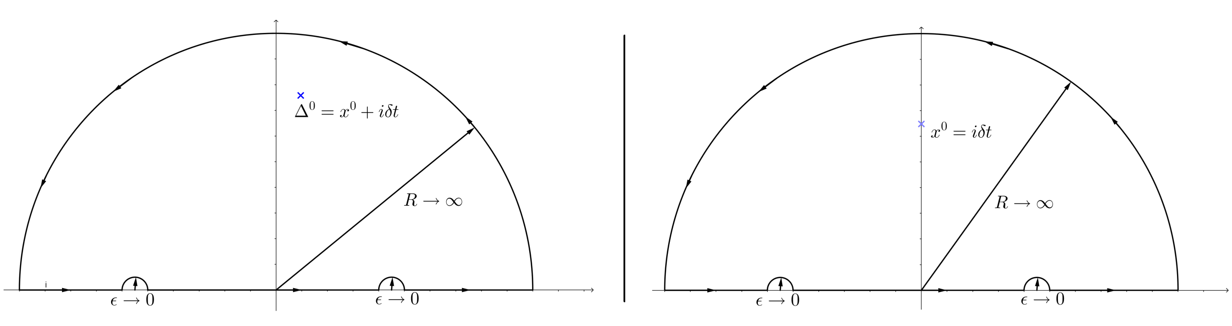


Figure D.1: In the $\varDelta^0$ integral (left diagram), the contour avoids the branch points on the real axis and picks up a residue at the simple pole at $\varDelta^0 = x^0 + i\delta t$. In the x^0 integral (right diagram), a similar contour is used. It now picks up a residue at the simple pole at $x^0 = i\delta t$.

The series expansion of this at large r_B gives the result in Eq. (3.18). We have also checked that this agrees with the result of numerically integrating Eq. (D.4).

The calculation of energy fluctuation in the null infinity limit parallels the calculation above. For concreteness, let's consider a scalar field, and take as our starting point Eq. (3.27). Inserting Eq. (3.26), and evaluating the $d^3\vec{x}$ integral, we get

$$\langle M^2 \rangle = \int dx^0 d\mathit{\Delta}^0 4\pi^2 d\mathit{\Delta}\, \mathrm{Vol}(r_1, r_2, \mathit{\Delta}) 8 \left(30\xi^2 - 10\xi + 1\right) \frac{3\vec{\mathit{\Delta}}^4 + 10\mathit{\Delta}_0^2 \vec{\mathit{\Delta}}^2 + 3\mathit{\Delta}_0^4}{\left(\mathit{\Delta}_0^2 - \vec{\mathit{\Delta}}^2\right)^6} w(x^0) w(\mathit{\Delta}^0). \tag{D.7}$$

Evaluating the $\mathit{\Delta}$ integral gives

$$\langle M^2 \rangle = -\frac{\left(30\xi^2 - 10\xi + 1\right) 128 \delta t^2 {r_1}^3 {r_2}^3 \left(-5(\mathit{\Delta}^0)^4 + 2(\mathit{\Delta}^0)^2 \left({r_1}^2 + {r_2}^2\right) + 3\left({r_1}^2 - {r_2}^2\right)^2\right)}{15\left(\delta t^2 + (x^0)^2\right)\left(\delta t^2 + (x^0 - \mathit{\Delta}^0)^2\right)(-\mathit{\Delta}^0 + r_1 - r_2)^3(-\mathit{\Delta}^0 + r_1 + r_2)^3(\mathit{\Delta}^0 + r_1 - r_2)^3(\mathit{\Delta}^0 + r_1 + r_2)^3} \tag{D.8}$$

Following the same contour prescription as before (see Figure D.1), the $\mathit{\Delta}^0$ integral picks up the residue at $\mathit{\Delta}^0 = t + ia$ and evaluates to

$$\begin{aligned}\langle M^2 \rangle = & \frac{128\left(30\xi^2 - 10\xi + 1\right)\pi\delta t((\alpha - 1)r_B + i\delta t\alpha)^3(-\alpha r_B + r_B + \alpha x^0)^3}{15(\alpha-1)^3(\alpha+1)^3(\delta t - ix^0)^5(\delta t + ix^0)(\delta t(\alpha+1) - i(\alpha-1)(2r_B - x^0))^3(\delta t(\alpha-1) - i(2(\alpha-1)r_B - (\alpha+1)x^0))^3} \\ & \times[\left(3\alpha^4 + 2\alpha^2 - 5\right)\delta t^2 + 2i\delta t\left(\left(3\alpha^4 + 5\right)x^0 - 2\alpha\left(3\alpha^3 - 3\alpha^2 + \alpha - 1\right)r_B\right) \\ & -(\alpha - 1)\left(4\left(3\alpha^3 - 3\alpha^2 + \alpha - 1\right)r_B^2 - 4\left(3\alpha^3 + \alpha\right)r_B x^0 + \left(3\alpha^3 + 3\alpha^2 + 5\alpha + 5\right)(x^0)^2\right)]\end{aligned} \tag{D.9}$$

A similar contour can be used for the x^0 integral now, which picks up a residue at $t = ia$, and gives the answer in Eq. (3.30).

Appendix E

Proof of Lemma 14

We now prove Lemma 14 by direct calculation.

Proof. We wish to show that

$$R|_{(0,s)} \equiv f_*\partial_r|_{(0,s)} = \exp_{K*}|_{(p,sS|_{(0,0)})}\,(\check{R}, s\tilde{R}) \tag{E.1}$$

and

$$S|_{(0,s)} \equiv f_*\partial_s|_{(0,s)} = \exp_{K*}|_{(p,sS|_{(0,0)})}\,(0, \tilde{S}), \tag{E.2}$$

where

$$\begin{aligned} f(r,s) &= \exp_{f(r,0)} sS|_{(r,0)} \\ &= \exp_K(f(r,0), sS|_{(r,0)}), \end{aligned} \tag{E.3}$$

as defined in Sec. 4.2.

Using the definition of the pushforward, we can write $f_*\partial_r|_{(0,s)}$ as the differential $\exp_{K*}$, associated with f in Eq. (E.3), evaluated along the tangent direction $sS|_{(0,0)}$,

$$\begin{aligned} R|_{(0,s)} &\equiv f_*\partial_r|_{(0,s)} \\ &= \exp_{K*}|_{(p,sS|_{(0,0)})}\,(\check{R}, s\phi_*(\partial_r S|_{(r,0)})|_{r=0}). \end{aligned} \tag{E.4}$$

In the second line, we used the definition of $\check{R}$ as the tangent to $\mu(r)$ at p, along with linearity of $\exp_{K*}$. We have again used the notation ϕ_* for the identity map between vectors in T_pM and their naturally associated counterparts in T_ST_pM.

Next, we must evaluate the derivative of S, $\phi_*(\partial_r S|_{(r,0)})|_{r=0} \in T_ST_pK^\perp$. Let us write $S|_{(r,0)}$ as an explicit function of both the parameter r along the path $\mu(r) \equiv f(r,0) \in K$ and the vector $S|_{(0,0)} + r\hat{R} \in T_pK^\perp$ that is normal parallel transported along μ from $\mu(0) = p$ to $\mu(r)$:

$$\begin{aligned} S|_{(r,0)} &= S(r, S|_{(0,0)} + r\hat{R})|_{r_1=r_2=r} \\ &\equiv S(r_1, r_2)|_{r_1=r_2=r}, \end{aligned} \tag{E.5}$$

so that the derivative in question can be written as $\phi_* \partial_r S(r, S|_{(0,0)} + r\hat{R})|_{r=0}$. Since $S(r_1, r_2)$ is defined by normal parallel transporting a particular vector $(S|_{(0,0)} + r\hat{R})$ in $T_p K^\perp$ to $\mu(r_1)$, its variation with respect to r_1 gives the normal part of the covariant derivative of S along μ, which vanishes, i.e., $\partial_{r_1} S(r_1, r_2) = 0$. Hence,

$$\begin{aligned} &\frac{\partial}{\partial r}\left[S(r_1, S|_{(0,0)} + r_2\hat{R})|_{r_1=r_2=r}\right] \\ &= \left[\frac{\partial}{\partial r_2} S(r, r_2)\right]_{r_2=r} \\ &= \hat{R}. \end{aligned} \tag{E.6}$$

Inputting this result into Eq. (E.4), we have

$$\begin{aligned} R|_{(0,s)} &= \exp_{K*}|_{(p, sS|_{(0,0)})}(\check{R}, s\phi_*\hat{R}) \\ &= \exp_{K*}|_{(p, sS|_{(0,0)})}(\check{R}, s\tilde{R}). \end{aligned} \tag{E.7}$$

We have thus derived the claimed formula for the Jacobi field stated in Eq. (E.1). The proof of Eq. (E.2) follows similarly. Neither $f(r, 0)$ or $S|_{(r,0)}$ depend on s. Therefore

$$\begin{aligned} S|_{(0,s)} &\equiv f_*\partial_s|_{(0,s)} \\ &= \partial_s \exp_K(f(0,0), sS|_{(0,0)}) \\ &= \exp_{K*}|_{(p, sS|_{(0,0)})}(0, \phi_* S|_{(0,0)}) \\ &= \exp_{K*}|_{(p, sS|_{(0,0)})}(0, \tilde{S}). \end{aligned} \tag{E.8}$$

This derivation of the Jacobi field and tangent vector completes the proof of Lemma 14. □

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